

XIII. Schwinger model for the angular momentum operator

Consider two independent bosonic modes described by annihilation/creation operator obeying the standard commutation relation:

$$[\hat{a}, \hat{a}^+] = 1, \quad [\hat{b}, \hat{b}^+] = 1, \quad [\hat{a}, \hat{b}] = 0, \quad [\hat{a}, \hat{b}^+] = 0$$

Then one can show that the operators

$$\hat{J}_x = \frac{\hat{a}^+ \hat{b} + \hat{b}^+ \hat{a}}{2}, \quad \hat{J}_y = \frac{\hat{a}^+ \hat{b} - \hat{b}^+ \hat{a}}{2i}, \quad \hat{J}_z = \frac{\hat{a}^+ \hat{a} - \hat{b}^+ \hat{b}}{2}$$

satisfy the commutation relation for the components of the angular momentum operator

$$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z, \quad [\hat{J}_y, \hat{J}_z] = i\hat{J}_x, \quad [\hat{J}_z, \hat{J}_x] = i\hat{J}_y.$$

Also $[\hat{J}_\ell, \hat{J}^2] = 0$, $\ell = x, y, z$, where

$$\hat{J}^2 \equiv \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = \left(\frac{\hat{N}}{2} + 1 \right) \frac{\hat{N}}{2}, \quad \hat{N} \equiv \hat{a}^+ \hat{a} + \hat{b}^+ \hat{b}$$

integer or

Eigenvalues N of \hat{N} are non-negative integers  the momentum $J = N/2$ is half-integer ≥ 0 .

Cyclic components

$$\hat{J}_{\pm 1} = \mp \frac{1}{\sqrt{2}} (\hat{J}_x \pm i \hat{J}_y), \quad \hat{J}_0 = \hat{J}_z$$

$$\hat{J}_{+1} = -\frac{\hat{a}^+ \hat{b}}{\sqrt{2}}$$

raises M by 1,

$$\hat{J}_{-1} = \frac{\hat{b}^+ \hat{a}}{\sqrt{2}}$$

lowers M by 1, where M is an eigenvalue of \hat{J}_0 .

Holstein–Primakoff transformation

Mapping of quantum ang.momentum to bosonic annihilation/creation operators.

Consider $|J, M = +J\rangle$ as a **vacuum** state and, respectively, $m = J - M$ as the number of excitations. Introduce formally bosonic operators $\hat{c}, \hat{c}^\dagger, [\hat{c}, \hat{c}^\dagger] = 1$

$$|J, M = J - m\rangle = (m!)^{-1/2} \hat{c}^{\dagger m} |\mathbf{vac}\rangle$$

Then, recalling the expression for the matrix elements of cyclic components of $\hat{\mathbf{J}}$ we obtain

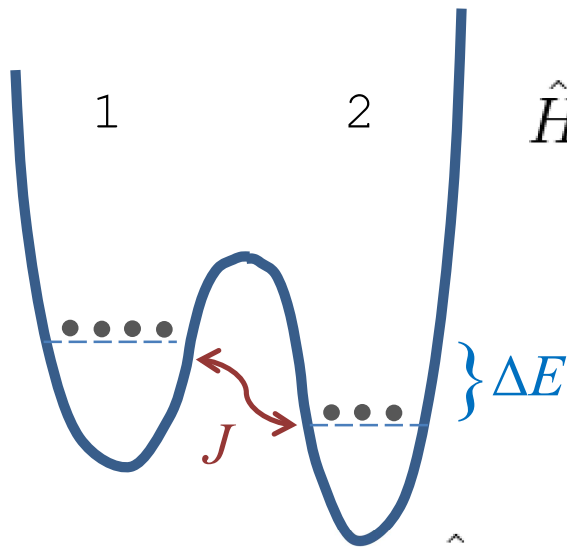
$$\hat{J}_0 = J - \hat{c}^\dagger \hat{c}, \quad \hat{J}_{+1} = -\sqrt{J} \sqrt{1 - \frac{\hat{c}^\dagger \hat{c}}{2J}} \hat{c}, \quad \hat{J}_{-1} = +\sqrt{J} \hat{c}^\dagger \sqrt{1 - \frac{\hat{c}^\dagger \hat{c}}{2J}}$$

This transformation is especially convenient for the small number of excitations, $m \ll J$,

where one can expand these expressions in Taylor series in $\frac{\hat{c}^\dagger \hat{c}}{2J}$.

XIV. Quantum models to be mapped on angular-momentum problems

(XIV.1) Two-mode Bose-Hubbard model (ultracold atoms in a double well potential – the simplest, 2-mode description)



$$\hat{H} = U_1 \hat{N}_1^2 + U_2 \hat{N}_2^2 + \frac{\Delta E}{2} (\hat{N}_1 - \hat{N}_2) - J (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)$$

$$\hat{N}_j = \hat{a}_j^\dagger \hat{a}_j, \quad j = 1, 2$$

$$\hat{N} = \hat{N}_1 + \hat{N}_2, \quad [\hat{N}, \hat{H}] = 0$$

$$\hat{S}_z = \frac{\hat{N}_1 - \hat{N}_2}{2}, \quad \hat{S}_x = \frac{\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1}{2}, \quad \hat{S}^2 = \frac{N}{2} \left(\frac{N}{2} + 1 \right)$$

$$\hat{H} = (U_1 + U_2) \frac{N^2}{4} + (U_1 + U_2) \hat{S}_z^2 + [\Delta E + N(U_1 - U_2)] \hat{S}_z - 2J \hat{S}_x$$

(XIV.2) Dicke model

A two-level system consisting of two states, ground $|g\rangle$ and excited $|e\rangle$, is formally equivalent to a (pseudo)spin $s = 1/2$.

The raising operator

$$\hat{\sigma}^+ = \hat{s}_x + i\hat{s}_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

transforms $|g\rangle$ into $|e\rangle$,

the lowering operator

$$\hat{\sigma}^- = \hat{s}_x - i\hat{s}_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

transforms $|e\rangle$ into $|g\rangle$.

The operator of the population difference

$$\hat{\sigma}_z = 2\hat{s}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hamiltonian of N two-level system with a single electromagnetic mode (practically, with a cavity mode). We denote the photon annihilation operator by \hat{a} .

If we assume that the atom-photon coupling constant is the same for all atoms, this

Hamiltonian reads as (we set Planck's constant $\hbar = 1$)

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \sum_{i=1}^N [\omega_0 \hat{S}_{z i} + 2g(\hat{a}^\dagger + \hat{a}) \hat{S}_{x i}]$$

Sum of individual spin operators yield the collective spin operator:

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{S}_z + 2g(\hat{a}^\dagger + \hat{a}) \hat{S}_x$$

Since the e.m.-mode is close to the resonance, $\omega \approx \omega_0$, we can use the rotating wave approximation (RWA):

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{S}_z + g(\hat{a}^\dagger \hat{S}^- + \hat{a} \hat{S}^+)$$

What is the integral of motion of this Hamiltonian?

Note: the same coupling constant for all atoms may be attained for a ring (running-wave) cavity; the phase factors $\exp(\mathbf{i}\mathbf{k}\mathbf{r}_j)$ for different atoms can be included into the definition of $|e\rangle$.

The use of the Holstein-Primakoff transformation reduces the Hamiltonian to one for two bosonic fields (atomic excitations and phonons). How this bosonic Hamiltonian looks if the number of at. excitations + the number of photons $\ll N$? In the case of small number of excitations and phonons write the Hamiltonian in the case of non-equal coupling constant (each atom possessing its own g_j).

But N pseudospins $s = 1/2$ may be summed in different ways.

If they form a fully symmetrized state, i.e., characterized by the Young diagram $\{N\}$, then we obtain max.possible collective spin $S = N/2$.

In a general case, for the Young diagram $\{N - m, m\}$, where $m \leq N/2$, we obtain $S = N/2 - m$.

In particular, for an even N and $m = N/2$ (the Young diagram consisting of two rows of the equal length) $S = 0$.

The rate Γ of photon emission into the cavity mode is proportional to $\langle \hat{S}^+ \hat{S}^- \rangle$

If (almost) all atoms are in the $|e\rangle$ state, $\langle \hat{S}_z \rangle \approx S$, then $\Gamma \propto S$.

When in the course of evolution, almost half of the atoms decayed into the state $|g\rangle$, i.e.,

when $\langle \hat{S}_z \rangle \approx 0$, we obtain $\Gamma \propto S^2$.

The states with $\{\lambda\} = \{N\}$ and, hence, $S = N/2$ are called Dicke states. They are characterized by the maximum possible photon emission rate

$$\langle \hat{S}_z \rangle \approx N/2 \quad \longrightarrow \quad \Gamma \propto N \quad \text{Atoms emit photons independently.}$$

$$\langle \hat{S}_z \rangle \approx 0 \quad \longrightarrow \quad \Gamma \propto N^2 \quad \text{Collective (enhanced) emission – superradiance.}$$

The opposite limit: states with $\{\lambda\} = \{N/2, N/2\}$ for even N and, hence, $S = 0$, do not emit into the cavity mode at all. **Do they emit into other modes (side modes)? Why?**

Single-electron qubits: besides the pseudospin, there is the spin of electron
 Wave function of N electrons:

$$|\Psi\rangle = \hat{A}|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle$$

 **antisymmetrization**

$$|\psi_j\rangle = |pseudospin_j\rangle \otimes |location_j\rangle \otimes |spin_j\rangle$$

$$\{\lambda_{loc}\}$$

$$\{\lambda_{spin}\}$$

IR tensor product (symmetric group S_N)

$$\{\lambda_{pseudospin}\}$$

$$\{\tilde{\lambda}_{pseudospin}\}$$

IR tensor product (symmetric group S_N)

$$\{1, 1, 1, \dots, 1\} \equiv \{1^N\}$$