Molecular symmetry group
II. Induced representations

We consider finite groups
A subgroup: $\mathcal{H} \subset \mathcal{G}$
Consider a representation T of $\mathcal{G}$: $g \mapsto \hat{T}(g)$ as a representation $T^{\mathcal{H}}$ of $\mathcal{H}$ only by restricting ourselves to $g \in \mathcal{H}$
Then it can be decomposed in IRs Q^j of $\mathcal{H}$ as

$$T^{\mathcal{H}} = \bigoplus_j m_j Q^{(j)}$$

$$m_j = \frac{1}{L_{\mathcal{H}}} \sum_{h \in \mathcal{H}} \chi_j^*(h) \chi_T(h)$$

$$\chi_j(h) = \sum_l Q_l l^{(j)}(h), \qquad \chi_T(h) = \sum_i T_{ii}(h)$$

$L_{\mathcal{H}}$ = order of $\mathcal{H}$

The opposite task: to extend a representation $\mathcal{Q}$ of $\mathcal{H}$ to $\mathcal{G}$, i.e., to construct an induced representation

First, recall that the ratio of the orders

$$Z = \frac{L_{\mathcal{G}}}{L_{\mathcal{H}}}$$

is an integer number

Expansion of the group in left cosets:

$\exists g_i \in \mathcal{G},\ i = 1, 2, \ldots, Z:$

$$\mathcal{G} = \bigcup_{i=1}^{Z} g_i \mathcal{H}, \qquad g_1 \equiv e$$

$$(g_i \mathcal{H}) \cap (g_{i'} \mathcal{H}) = 0, \quad i \neq i'$$

Let the representation Q of $\mathcal{H}$ be defined over the vector space $V_{\mathcal{H}}$.
The latter can be considered as a subset of a larger vector space, where we can define $g_i V_{\mathcal{H}}$ for all $i = 1, 2, \ldots, Z$ $(g_1 \equiv e)$
Then the induced representation is defined over

$$V_{\mathcal{G}} = \bigcup_{i=1}^{Z} g_i V_{\mathcal{H}}$$

Consider a group operation g acting on a $\mathbf{x} \in g_i V_{\mathcal{H}}$,
i.e., $\mathbf{x} = g_i \mathbf{y}, \quad \mathbf{y} \in V_{\mathcal{H}}$
We find first, to which coset $g g_i$ belongs, i.e., we find g_j such that

$$g g_i = g_j h, \;\; h \in \mathcal{H}$$

$$g g_i \mathbf{y} = g_j h \mathbf{y} = g_j \mathbf{y}', \quad \mathbf{y}' \in V_{\mathcal{H}} \;\Rightarrow\; g_j \mathbf{y}' \in g_j V_{\mathcal{H}}$$

This allows us to calculate all the matrix elements for all operators $\hat{T}(g)$ for the induced representation T.

Frobenius formula

expressing the character of the induced respresentation T via the character of Q:

$$\chi_T(g) = \sum_{\substack{g_i \in \mathcal{G}/\mathcal{H} \\ g_i^{-1} g g_i \in \mathcal{H}}} \chi_Q(g_i^{-1} g g_i) = \frac{1}{L_{\mathcal{H}}} \sum_{\substack{f \in \mathcal{G} \\ f^{-1} g f \in \mathcal{H}}} \chi_Q(f^{-1} g f)$$

Expansion of T in IRs T_j of $\mathcal{G}$ is given by the character formula