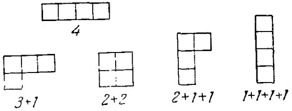
# Representations of the symmetric group

# Young diagrams

Integer partition:

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_m, \quad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$$

corresponds to a Young diagram with $m$ rows of resp. lengths $\lambda_i$




Young tableau: filled with numbers

Permutation of the numbered boxes.
We denote by $p$ permutations that interchange boxes within their respective rows and by $q$ permutations that interchange boxes within their respective columns.
Function on the symmetric group ($s \in S_n$):

$$\varphi(s) = \begin{cases} \omega_q, & s = qp \\ 0, & s \neq qp \end{cases} , \quad \omega_q = \begin{cases} 1, & q \text{ even} \\ -1, & q \text{ odd} \end{cases}$$

Consider $s \in S_n$ as the argument of a function and all $t \in S_n$ as the parameters of the function, such that

$$\varphi_t(s) = \varphi(st)$$

and a linear space $\mathcal{L}$ spanned by these functions. For all $r \in S_n$ we define operators

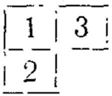
$$\hat{\tau}(r)\varphi_t(s) = \varphi_t(sr)$$

$\mathcal{L}$ invariant with resp. to these operators, since

$$\varphi_t(sr) = \varphi(srt) = \varphi_{rt}(s)$$

Obviously, $\tau$ is a representation, since $\tau(r_1)\tau(r_2) = \tau(r_1 r_2)$. Moreover, it is an IR. Young diagrams $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ specify all IRs of $S_n$.

Example: explicit construction of the IR of $S_3$ corresponding to $\{2, 1\}$




$$p: \quad e, \ (1,3)$$
$$q: \quad e, \ (1,2)$$

Products $qp$: $\quad e, \ (1,3), \ (1,2), \ (1,2)(1,3) = (1,3,2)$

| $s$ | $e$ | (12) | (23) | (13) | (123) | (132) |
|---|---|---|---|---|---|---|
| $\varphi(s)$ | 1 | $-1$ | 0 | 1 | 0 | $-1$ |

| | $e$ | (12) | (23) | (13) | (123) | (132) |
|---|---|---|---|---|---|---|
| $\varphi_{12}(s)$ | $-1$ | 1 | $-1$ | 0 | 1 | 0 |
| $\varphi_{23}(s)$ | 0 | 0 | 1 | $-1$ | $-1$ | 1 |
| $\varphi_{13}(s)$ | 1 | $-1$ | 0 | 1 | 0 | $-1$ |
| $\varphi_{123}(s)$ | 0 | 0 | 1 | $-1$ | $-1$ | 1 |
| $\varphi_{132}(s)$ | $-1$ | 1 | $-1$ | 0 | 1 | 0 |

Only two functions are linearly independent.

$$\varphi 13 = \varphi, \quad \varphi 132 = \varphi 12, \quad \varphi 23 = -\varphi - \varphi 12, \quad \varphi 123 = -\varphi - \varphi 12$$

$$\tau(12)\,\varphi_{12} = \varphi_{(12)\ (12)} = \varphi,$$

$$\tau(23)\,\varphi_{12} = \varphi_{(23)\ (12)} = \varphi_{132} = \varphi_{12},$$

$$\tau(13)\,\varphi_{12} = -\varphi - \varphi_{12},$$

$$\tau(123)\,\varphi_{12} = \varphi,$$

$$\tau(132)\,\varphi_{12} = -\varphi_{1} - \varphi_{12},$$

$$e \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (12) \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (23) \sim \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$(13) \sim \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad (123) \sim \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad (132) \sim \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Characters of the representation $\{2, 1\}$: $\chi(e) = 2$;
$\chi(12) = 0$ for permutations of a single pair of objects only;
$\chi(123) = -1$ for cyclic permutations of three objects.

# Explicit construction of functions of 3 variables transforming according to IRs of $S_3$

Three identical particles characterized by variables $\alpha_1,\ \alpha_2,\ \alpha_3$ (co-ordinates or spins); three single-particle states $\psi_1,\ \psi_2,\ \psi_3$.
Total number of all possible 3-particle states in this basis: $3^3 = 27$

| Symmetric functions (IR ={3}): | # of functions |
|---|---|
| $\psi_j(\alpha_1)\psi_j(\alpha_2)\psi_j(\alpha_3), \qquad j = 1, 2, 3$ | 3 |
| $\frac{1}{\sqrt{3}}[\psi_j(\alpha_1)\psi_j(\alpha_2)\psi_k(\alpha_3) + \psi_j(\alpha_1)\psi_k(\alpha_2)\psi_j(\alpha_3) + \psi_k(\alpha_1)\psi_j(\alpha_2)\psi_j(\alpha_3)], \quad j = 1, 2, 3,\ k \neq j$ | $3 \times 2 = 6$ |
| $\frac{1}{\sqrt{6}}\sum_P \hat{P}_{(i_1 i_2 i_3)}\psi_1(\alpha_{i_1})\psi_2(\alpha_{i_2})\psi_3(\alpha_{i_3})$ | 1 |
| | TOTAL: 10 |

| Antisymmetric function (IR ={1, 1, 1}): | # of functions = 1 |
|---|---|
| $\frac{1}{\sqrt{6}}\sum_P (-1)^P \hat{P}_{(i_1 i_2 i_3)}\psi_1(\alpha_{i_1})\psi_2(\alpha_{i_2})\psi_3(\alpha_{i_3})$ | |

The remaining 16 functions are transformed according to IR ={2, 1}

$3 \times 2 = 6$ ways to choose $\psi_j\psi_j\psi_k, \quad j = 1, 2, 3, \ k \neq j.$

$2 \times 6$ linear combinations orthogonal to

$$\frac{1}{\sqrt{3}}[\psi_j(\alpha_1)\psi_j(\alpha_2)\psi_k(\alpha_3) + \psi_j(\alpha_1)\psi_k(\alpha_2)\psi_j(\alpha_3) + \psi_k(\alpha_1)\psi_j(\alpha_2)\psi_j(\alpha_3)] \equiv \frac{1}{\sqrt{3}}[\Psi_{jjk} + \Psi_{jkj} + \Psi_{kjj}] \ :$$

$$\Psi_\pm(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{\sqrt{3}}\left[e^{\pm 2\pi i/3}\psi_j(\alpha_1)\psi_j(\alpha_2)\psi_k(\alpha_3) + \psi_j(\alpha_1)\psi_k(\alpha_2)\psi_j(\alpha_3) + e^{\mp 2\pi i/3}\psi_k(\alpha_1)\psi_j(\alpha_2)\psi_j(\alpha_3)\right] =$$

$$= \frac{1}{\sqrt{3}}\left[e^{\pm 2\pi i/3}\Psi_{jjk} + \Psi_{jkj} + e^{\mp 2\pi i/3}\Psi_{kjj}\right]$$

$$\hat{\tau}(g)\Psi_\mu = \sum_{\mu'=+,-}\Psi_{\mu'}\tau_{\mu'\mu}(g), \qquad \mu = +, -$$

# Transformation of $\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}$

$$\tau(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau(12) = \begin{pmatrix} 0 & e^{2\pi i/3} \\ e^{-2\pi i/3} & 0 \end{pmatrix},$$

$$\tau(23) = \begin{pmatrix} 0 & e^{-2\pi i/3} \\ e^{2\pi i/3} & 0 \end{pmatrix}, \quad \tau(13) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\tau(123) = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{pmatrix}, \quad \tau(132) = \begin{pmatrix} e^{-2\pi i/3} & 0 \\ 0 & e^{2\pi i/3} \end{pmatrix}.$$

$$\chi(e) = 2, \qquad \chi(12) = 0, \qquad \chi(123) = -1.$$

$$\tau(123) = \tau(23)\tau(12), \qquad \tau(132) = \tau(23)\tau(13).$$

$$\tau^{-1}(g) = \tau^{\dagger}(g)$$

# Three different single-particle states and $\{2,1\}$ permutation symmetry

We construct $\Psi_\pm$ from $\psi_1\psi_1\psi_3$ and apply to them an operator $\hat{\mathcal{F}}_{2\leftarrow 1} = \sum_{n=1}^{3} \hat{F}^{(n)}_{2\leftarrow 1}$ defined via

$$\hat{F}^{(n)}_{2\leftarrow 1}\,\psi_1(\alpha_n) = \psi_2(\alpha_n), \quad \hat{F}^{(n)}_{2\leftarrow 1}\psi_{2,3}(\alpha_n) = 0, \;\; n = 1,2,3,$$

which is fully symmetric against permutations of the particles.
After normalization to 1:

$$\Phi^{[113]}_\pm = \frac{1}{\sqrt{2}}\hat{\mathcal{F}}_{2\leftarrow 1}\,\Psi^{[113]}_\pm = \frac{1}{\sqrt{6}}\left[e^{\pm 2\pi i/3}\Psi_{123} + e^{\pm 2\pi i/3}\Psi_{213} + \Psi_{132} + \Psi_{231} + e^{\mp 2\pi i/3}\Psi_{312} + e^{\mp 2\pi i/3}\Psi_{321}\right].$$

$\forall g \in S_3 : \;\hat{\tau}(g)\hat{\mathcal{F}}_{2\leftarrow 1} = \hat{\mathcal{F}}_{2\leftarrow 1}\hat{\tau}(g) \;\Rightarrow$ the matrix $\tau(g)$ has in the $\Phi_\pm$ basis the same form as in the $\Psi_\pm$ basis.
Another pair of lin. independent functions: $\Phi^{[112]}_\pm = \frac{1}{\sqrt{2}}\hat{\mathcal{F}}_{3\leftarrow 1}\,\Psi^{[112]}_\pm$.

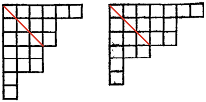
# Bosonic and fermionic wave functions

$$\hat{P}\,\Psi_B(x_1,\ldots,x_n;\,\sigma_1,\ldots,\sigma_n) = \Psi_B(x_1,\ldots,x_n;\,\sigma_1,\ldots,\sigma_n)$$

$$\hat{P}\,\Psi_F(x_1,\ldots,x_n;\,\sigma_1,\ldots,\sigma_n) = (-1)^P\,\Psi_F(x_1,\ldots,x_n;\,\sigma_1,\ldots,\sigma_n)$$

If the co-ordinate part transforms according to IR of $S_n$ given by a certain Young diagram, then the spin part transforms according to

- the same IR for bosons;
- the IR corresponding to a transposed Young diagram for fermions.



Dimensions of representations for a Young diagram and its transpose are the same.

For a given unitary representation of dimension $s$:

$$\Psi_B(x_1,\ldots,x_n;\,\sigma_1,\ldots,\sigma_n) = \sum_{j=1}^{s} R_j(x_1,\ldots,x_n) W_j \sigma_1,\ldots,\sigma_n),$$

$$\Psi_F(x_1,\ldots,x_n;\,\sigma_1,\ldots,\sigma_n) = \sum_{j=1}^{s} R_j(x_1,\ldots,x_n) \tilde{W}_j \sigma_1,\ldots,\sigma_n),$$

where the permutation transformations $P$ are given by

$$\hat{\tau}(P) R_j = \sum_{k=1}^{s} R_k \tau_{kj}(P), \qquad \hat{\tau}(P) W_j = \sum_{k=1}^{s} W_k \tau_{kj}^{*}(P),$$

$$\hat{\tau}(P) \tilde{W}_j = (-1)^P \sum_{k=1}^{s} \tilde{W}_k \tau_{kj}^{*}(P)$$

# Dimensions for IRs of $S_n$

Young tableau: a Young diagram ($n$ boxes) filled with integer numbers 1,2, ... , $n$.
Standard Young tableau: numbers in each row and in each column are placed in the increasing order.
Example: all standard Young tableaux for $\{\lambda\}=\{3,2\}$.

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | |

| | | |
|---|---|---|
| 1 | 2 | 4 |
| 3 | 5 | |

| | | |
|---|---|---|
| 1 | 2 | 5 |
| 3 | 4 | |

| | | |
|---|---|---|
| 1 | 3 | 4 |
| 2 | 5 | |

| | | |
|---|---|---|
| 1 | 3 | 5 |
| 2 | 4 | |

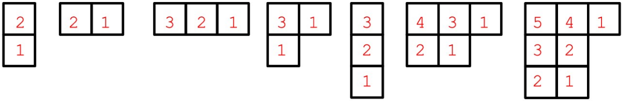
Dimension of IR = $\{\lambda\}$ is equal to the number of standard Young tableaux.

# Hook length formula

Hook length $h(j)$ of the $j$th box of a Young diagram:

$h(j) =$ number of boxes to the right in the row +
+ number of boxes below in the column + 1

Hook length for some $\{\lambda\}$




An easier way to calculate $d_{\{\lambda\}}$ is given by a theorem:

$$d_{\{\lambda\}} = \frac{n!}{\prod_{j=1}^{n} h(j)}$$

$$\sum_{\{\lambda\}} d_{\{\lambda\}}^2 = n!$$

# For three particles

A non-trivial case: $\{2, 1\}$; this Young diagram is its own transpose.
Co-ordinate basis:

$$R_1 = \Psi_+(x_1, x_2, x_3), \qquad R_2 = \Psi_-(x_1, x_2, x_3).$$

Spin basis

$$W_1 = \Psi_-(\sigma_1, \sigma_2, \sigma_3), \quad W_2 = \Psi_+(\sigma_1, \sigma_2, \sigma_3) \quad \text{for bosons,}$$

$$\tilde{W}_1 = \Psi_-(\sigma_1, \sigma_2, \sigma_3), \quad \tilde{W}_2 = -\Psi_+(\sigma_1, \sigma_2, \sigma_3) \quad \text{for fermions.}$$

# Total spin and permutation symmetry

Single-particle spin $s$; total spin $S$ and its projection $S_z$.
Both $\hat{S}^2$ and $\hat{S}_z$ commute with spin permutations. Therefore, $|S,\ S_z\rangle$ have also additional quantum number, characterizing their symmetry against permutations.

Total spin for three s=1/2 particles: $1/2 + 1/2 \rightarrow 0, 1$.
$0 + 1/2 \rightarrow 1/2$, $\quad 1 + 1/2 \rightarrow 1/2, 3/2$.

Total spin for three s=1 particles: $1 + 1 \rightarrow 0, 1, 2$.
$0 + 1 \rightarrow 1$, $\quad 1 + 1 \rightarrow 0, 1, 2$, $\quad 2 + 1 \rightarrow 1, 2, 3$.

Total spin $S$

| $\{\lambda\}$ | $s = 0$ | $s = \frac{1}{2}$ | $s = 1$ |
|---|---|---|---|
| $\{3\}$ | 0 | $\frac{3}{2}$ | 3, 1 |
| $\{2, 1\}$ | – | $\frac{1}{2}, \frac{1}{2}$ | 2, 2, 1, 1 |
| $\{1, 1, 1\}$ | – | – | 0 |