Theory of characters

Character of a representation

For a given representation T of a dimension s for all $g \in G$:

$$\chi(g) = \sum_{i=1}^{s} T_{ii}(g)$$

Properties of characters:

1. Characters of equivalent representations are equal.
2. For IRs $(\chi^{(\alpha)}, \chi^{(\beta)}) = \delta_{\alpha\beta}$,
   where $T^{(\alpha)}$ and $T^{(\beta)}$ are not equivalent for $\alpha \neq \beta$.
3. If $T = T^{(1)} + \cdots + T^{(J)} = \sum_\nu m_\nu T^{(\nu)}$ then
   $\chi(g) = \sum_\nu m_\nu \chi^{(\nu)}(g)$.
4. The numbers $m_\nu = (\chi, \chi^{(\nu)})$ determine T up to equivalence.

5. Irreducibility criterion: T is a IR $\Leftrightarrow$ $(\chi, \chi) = 1$.

   Proof: $(\chi, \chi) = \left(\sum_\nu m_{\nu'} \chi^{(\nu')}, \sum_\nu m_\nu \chi^{(\nu)}\right) = \sum_\nu m_\nu^2$.

6. For conjugate elements: $\chi(g) = \chi(xgx^{-1})$.

   Proof: $\chi(g) = \sum_i T_{ii}(xgx^{-1}) = \sum_{i,j,k} T_{ij}(x) T_{jk}(g) T_{ki}(x^{-1}) =$
   $= \sum_{j,k} \delta_{jk} T_{jk}(g) = \sum_j T_{jj}(g) = \chi(g)$.

   I.e., characters of the elements of the same class are equal.

7. According the property **(6)**, we can write characters as functions of classes, i.e., if $g \in K_g$, then we can write instead

$$\chi = \chi(K_g).$$

   Characters of all IRs comprise a complete set of functions defined on the classes of conjugate elements. This follows from the completeness and orthogonality of matrix elements of IRs.

The number of lin. independent functions on a set of q classes is q; consider, e.g., $f(K_g) = \delta_{g,g'}$, $g' = 1, 2, \ldots, q$.
The number of non-equivalent IRs is equal to the number of classes.

If ρ_g is the number of elements in the class K_g, then

$$\sum_{\nu=1}^{q} \chi^{(\nu)}(K_g)\chi^{(\nu)\,*}(K_{g'}) = \frac{1}{\rho_g}\delta_{gg'}$$

(orthogonality relation for characters).

Product of representation

If T_1, T_2, and T_3 are representations of the same group and $\chi^{(3)}(g) = \chi^{(1)}(g)\chi^{(2)}(g)$, then T_3 is called a product of T_1 and T_2,

$$T_3 = T_1 \times T_2 = T_2 \times T_1 .$$

A product can be constructed for any two representations.
Let T_j be defined in $\mathcal{L}_j$, $\dim \mathcal{L}_j = s_j, \ j = 1, 2.$
For basis vectors:

$$\hat{T}_j(g)e_k^{(j)} = \sum_{i=1}^{s_j} e_i^{(j)} T_{ik}^{(j)}(g), \qquad j = 1, 2.$$

Let us define $\mathcal{L}$ as a linear space of $\dim \mathcal{L} = s_1 s_2$ and the basis $e_i^{(1)} e_k^{(2)}, \ i = 1, 2, \ldots s_1, \ k = 1, 2, \ldots s_2.$

We define a linear operator $\hat{T}(g)$ via its action on the basis vectors:

$$\hat{T}(g)e_i^{(1)}e_k^{(2)} = \sum_{i'=1}^{s_1}\sum_{k'=1}^{s_2} e_{i'}^{(1)}T_{i'i}^{(1)}(g)e_{k'}^{(2)}T_{k'k}^{(2)}(g).$$

Obviously, operator $\hat{T}(g)$

1. Yield a representation, since

$$\hat{T}(g_1g_2) = \hat{T}(g_1)\hat{T}(g_2)$$

2. That satisfies the definition of product $T = T_1 \times T_2$:

$$\chi(g) = \chi^{(1)}(g)\chi^{(2)}(g)$$

Consider a space $\mathcal{A}$ of linear operators that $\mathcal{L}_1 \mapsto \mathcal{L}_2$.
$\dim \mathcal{A} = s_1 s_2$.
We define a linear operator $\hat{T}(g)$ such that

$$\forall \hat{A} \in \mathcal{A} : \quad \hat{T}(g)\hat{A} = \hat{T}_2(g)\hat{A}\hat{T}_1(g^{-1}).$$

It is easy to prove that operators $\hat{T}(g)$ yield a representation and

$$\chi(g) = \chi^{(2)}(g)\chi^{(1)}(g^{-1})$$

If we choose the unit representation as T_2, then $\chi(g) = \chi^{(1)}(g^{-1})$.
For every representation T, a conjugate representation $\tilde{T}$ exists, such that

$$\tilde{\chi}(g) = \chi(g^{-1})$$

$$\dim T = \dim \tilde{T}$$

If τ is an IR, then $\tilde{\tau}$ is also an IR.

Theorem

If τ_1 and τ_2 are two IRs, then
$\tilde{\tau}_1 \times \tau_2$ contains the unit representation $\Leftrightarrow \tau_1 \sim \tau_2$.

Proof: using the Shur lemma.

$\tilde{\tau} \times \tau$ contains the unit representation one and only time.

For T and $\tilde{T}$: $\quad \tilde{T}_{ik}(g) = T_{ki}(g^{-1})$.