Representations of groups (continued)

Reducible and irreducible representations

A representation T of a group G in a linear space $\mathcal{L}$ is called <u>reducible</u>, if

$$\exists \mathcal{L}_1 \subset \mathcal{L},\ \mathcal{L}_1 \neq \{\mathbf{0}\} : \ \forall g \in G,\ \forall \mathbf{x} \in \mathcal{L}_1 : \ \hat{T}(g)\mathbf{x} \in \mathcal{L}_1$$

$\mathcal{L}_1$ is invariant with respect to all $\hat{T}(g)$.

3 A representation T of a group G in a linear space $\mathcal{L}$ is called <u>irreducible</u> (IR), if $\mathcal{L}$ contains no nontrivial subspace invariant with respect to $\hat{T}(g)$.

<u>Examples:</u>

1. All one-dimensional representations are irreducible.
2. Consider the group D_n and its representation in a 3D vector space. The latter can be divided into two subspaces: (a) vectors parallel to the z-axis, (b) vectors in a horizontal plane. I.e., we explicitly show that the representation mentioned above is reducible.

Induced representations

Assume that T in $\mathcal{L}$ is a reducible representation;
a non-trivial $\mathcal{L}_1 \subset \mathcal{L}$ is invariant with resp. to all $\hat{T}(g)$;
we can define an operator in a linear space of a smaller dimension:

$$\forall g \in G,\ \forall \mathbf{x} \in \mathcal{L}_1 : \quad \hat{T}_1(g)\mathbf{x} = \hat{T}(g)\mathbf{x}$$

Operator $\hat{T}(g)$ induces $\hat{T}_1(g)$ in a linear space $\mathcal{L}_1$;
The reducible repr. T induces T_1 in a linear space $\mathcal{L}_1$.

If T is unitary then T_1 is also unitary.

The combination of all irreducible representations induced by a reducible (unitary) T fully characterizes T.

Theorem

If T is a reducible unitary representation of G in $\mathcal{L}$ and $\mathcal{L}_1 \subset \mathcal{L}$ is invariant with resp. to all $\hat{T}(g)$ then $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$
[strictly speaking, $\mathcal{L}_2 = (\mathcal{L} \setminus \mathcal{L}_1) \cup \{\mathbf{0}\}$]
is also invariant with resp. to $\hat{T}(g)$.

Let $\mathbf{x} \in \mathcal{L}_1$ and $\mathbf{y} \in \mathcal{L}_2$.
Then $\hat{T}^{-1}(g)\mathbf{x} \in \mathcal{L}_1$.
Then the vectors $\hat{T}^{-1}(g)\mathbf{x}$ and $\mathbf{y}$ are orthogonal:

$$\langle \mathbf{x} | \hat{T}(g) | \mathbf{y} \rangle = 0$$

i.e., $\hat{T}(g)\mathbf{y}$ orthogonal to all vectors $\mathbf{x}$ from $\mathcal{L}_1$,
i.e., $\hat{T}(g) : \ \mathcal{L}_2 \mapsto \mathcal{L}_2$ (invariance proven).

$\mathcal{L}$ is split into two subspaces: $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ If $\hat{T}(g)$ induces $\hat{T}_1(g)$ and $\hat{T}_2(g)$ in respective subspaces and $\mathbf{x} \in \mathcal{L}_1$, $\mathbf{y} \in \mathcal{L}_2$, then for $\mathbf{z} = \mathbf{x} + \mathbf{y}$

$$\hat{T}(g)\mathbf{z} = \hat{T}_1(g)\mathbf{x} + \hat{T}_2(g)\mathbf{y}$$

If we know the representations T_1 and T_2 then we know T.
We call T a sum of T_1 and T_2.
If one of T_1, T_2 is reducible, we can represent it as a sum of two new representations etc.
Finally, we have a splitting:
Linear space is split into mutually orthogonal subspaces

$$\mathcal{L} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \ldots \mathcal{L}^{(N)}$$

The reducible representation is a sum of IRs:

$$T = T^{(1)} + T^{(2)} + \ldots T^{(N)}$$

If in $T = T^{(1)} + T^{(2)} + \ldots T^{(N)}$ there are m_l IRs, which are equivalent to a certain IR $T^{(l)}$, then we say that $T^{(l)}$ is contained in T m_l times. If none of $T^{(1)}, T^{(2)}, \ldots, T^{(N)}$ is equivalent to an IR $T^{(0)}$ of the same group G, then we say that $T^{(0)}$ is not contained in T.

Any reducible unitary (finite-dimensional) representation can be split into IRs and constructed as their sum.

Obviously, if the elements of the sum of representations are equivalent, $T_1 \sim T_1'$ and $T_2 \sim T_2'$, then

$$T_1 + T_2 \sim T_1' + T_2'$$

Lemma

Assume that τ is an irreducible representation of G. Then

$$\forall g \in G : \hat{A}\hat{\tau}(g) = \hat{\tau}(g)\hat{A} \quad \Rightarrow \quad \hat{A} = \lambda\hat{E},$$

where $\hat{E}$ is the identity operator in the respective $\mathcal{L}_\tau$.

<u>Proof</u>: $\hat{A}$ is a linear operator, therefore it has at least one eigenvalue (we denote it by λ). There is a subspace $\mathcal{L}_1$:

$$\mathbf{x} \in \mathcal{L}_1 \Rightarrow \hat{A}\mathbf{x} = \lambda\mathbf{x}.$$

Obviously, $\mathcal{L}_1 \neq \varnothing$, otherwise λ is not an eigenvalue.
$\forall \mathbf{x} \in \mathcal{L}_1 : \hat{A}\hat{\tau}(g)\mathbf{x} = \hat{\tau}(g)\hat{A}\mathbf{x} = \lambda\hat{\tau}(g)\mathbf{x}$,
i.e., $\hat{\tau}(g)\mathbf{x} \in \mathcal{L}_1 \Rightarrow \mathcal{L}_1$ is invariant with resp. to all $\hat{\tau}(g)$.
By assumption, τ is an IR, hence, $\mathcal{L}_1 = \mathcal{L}_\tau$,

$$\forall \in \mathcal{L}_\tau : \hat{A}\mathbf{x} = \lambda\mathbf{x} \quad \Rightarrow \quad \hat{A} = \lambda\hat{E}$$

Functions generated by a representation

Let T be a (reducible or irreducible) representation of G in $\mathcal{L}$, $\dim \mathcal{L} = s$. Choose an orthogonal and normalized basis $\mathbf{e}_i$, $i = 1, 2, \ldots, s$.
The matrix elements $T_{ik}(g)$ defined via

$$\hat{T}(g)\mathbf{e}_k = \sum_{i=1}^{s} T_{ik}(g)\mathbf{e}_i, \qquad T_{ik}(g) = \mathbf{e}_i^* T_{ik}(g)\mathbf{e}_k,$$

yield s^2 functions of g. Obviously,

$$T_{ik}(gh) = \sum_{l=1}^{s} T_{il}(g)T_{lk}(h)$$

Definition of a scalar product of the functions on a group:

$$(\varphi, \psi) = \frac{1}{N} \sum_{g \in G} \varphi(g) \psi^*(g)$$

Functions φ_1 and φ_2 are orthogonal $\Leftrightarrow$ $(\varphi_1, \varphi_2) = 0$.
A similar definition can be given for infinite groups, if the averaging functional $\mathcal{M}$ exists:

$$(\varphi, \psi) = \mathcal{M}\{\varphi, \psi^*\}$$

1st orthogonality theorem

A unitary IR τ of dimension s generates s^2 mutually orthogonal functions $\tau_{ik}(g),\ i,k = 1,2,3,\ldots,s,$

$$(\tau_{ik}, \tau_{i'k'}) = \frac{1}{s}\delta_{ii'}\delta_{kk'}$$

Proof: Let $\hat{B}$ be a lin.operator in the same $\mathcal{L}$, where τ is defined.

$$\hat{A} \equiv \frac{1}{N}\sum_{h\in G}\hat{\tau}(h)\hat{B}\hat{\tau}(h^{-1})$$

$$\hat{\tau}(g)\hat{A} = \frac{1}{N}\sum_{h\in G}\hat{\tau}(g)\hat{\tau}(h)\hat{B}\hat{\tau}(h^{-1}) = \frac{1}{N}\sum_{h\in G}\hat{\tau}(gh)\hat{B}\hat{\tau}(h^{-1})$$

We replace sum over all h via sum over $h' = gh$:

$$\hat{\tau}(g)\hat{A} = \frac{1}{N}\sum_{h'\in G}\hat{\tau}(h')\hat{B}\hat{\tau}(h'^{-1}g) = \frac{1}{N}\sum_{h'\in G}\hat{\tau}(h')\hat{B}\hat{\tau}(h'^{-1})\hat{\tau}(g) = \hat{A}\hat{\tau}(g)$$

$\hat{A}$ commutes with all $\hat{\tau}(g)$, hence, $\hat{A} = \lambda\hat{E}$.
More precisely,

$$\frac{1}{N}\sum_{h\in G}\hat{\tau}(h)\hat{B}\hat{\tau}(h^{-1}) = \lambda(B)\hat{E}$$

$$\frac{1}{N}\sum_{h\in G}\sum_{jl}\tau_{ij}(h)B_{jl}\tau_{lk}(h^{-1}) = \lambda(B)\delta_{ik}$$

$\lambda(B)$ is yet to be determined; in a general case (for $s > 1$), it is not an eigenvalue of $\hat{B}$.
By assumption, $\hat{\tau}(g)$ is unitary (we can always choose a unitary IR), therefore, $\tau_{lk}(h^{-1}) = [\hat{\tau}^{-1}(h)]_{lk} = [\hat{\tau}^{\dagger}(h)]_{lk} = \tau^*_{kl}(h)$

$$\frac{1}{N}\sum_{h\in G}\sum_{jl}\tau_{ij}(h)B_{jl}\tau^*_{kl}(h) = \sum_{jl}(\tau_{ij},\tau_{kl})B_{jl} = \lambda(B)\delta_{ik}$$

Choose $\hat{B}$ such that $B_{jl} = \delta_{ji'}\delta_{lk'}$, i.e., $\hat{B}$ is a linear operator mapping $\mathbf{e}_k$ to $\mathbf{e}_{i'}$ if $k = k'$ and to $\mathbf{0}$ otherwise.
For a given pair $i',\ k'$:

$$(\tau_{ii'},\tau_{kk'}) = \lambda_{i'k'}\delta_{ik}$$

Taking trace: $\sum_{i=1}^{s}\delta_{ii} = s$

$$\sum_{i}(\tau_{ii'},\tau_{ik'}) = \frac{1}{N}\sum_{h\in G}\sum_{i}\tau_{k'i}(h^{-1})\tau_{ii'}(h) = \tau_{k'i'}(e) = \delta_{k'i'}$$

$$\lambda_{i'k'} = \frac{1}{s}\delta_{i'k'} \quad \Rightarrow \quad (\tau_{ii'},\tau_{kk'}) = \frac{1}{s}\delta_{ik}\delta_{i'k'}$$

Schur lemma

If $\tau^{(1)}$ and $\tau^{(2)}$ are two non-equivalent IRs of G acting in $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and $\hat{A}: \quad \mathcal{L}_2 \mapsto \mathcal{L}_1$,

$$\forall g \in G \quad \hat{\tau}^{(1)}(g)\hat{A} = \hat{A}\hat{\tau}^{(2)}(g),$$

then $\hat{A} = 0$.

The main idea of the proof: if $\hat{A}$ is non-zero, then these two representations are either equivalent or reducible.

2nd orthogonality theorem

If $\tau^{(1)}$ and $\tau^{(2)}$ are two non-equivalent IRs of G then

$$(\tau^{(1)}_{ik}, \tau^{(2)}_{\alpha\beta}) = 0.$$

Proof: Let $\hat{B}$ be an operator $\mathcal{L}_2 \mapsto \mathcal{L}_1$.

$$\hat{A} = \frac{1}{N} \sum_{h \in G} \hat{\tau}^{(1)}(h) \hat{B} \hat{\tau}^{(2)}(h^{-1})$$

satisfies $\hat{\tau}^{(1)}(g)\hat{A} = \hat{A}\hat{\tau}^{(2)}(g)$ for all $g \in G \implies \hat{A} = 0$ (Schur lemma).
The rest of the proof is analogous to the 1st orthogonality theorem.

Completeness theorem

The set of functions $\tau_{ik}^{(a)}(g)$, $a = 1, 2, \ldots, q \quad ; i, k = 1, 2, \ldots, s_a$, generated by all non-equivalent IRs of G is complete, that is, any function $\varphi(g)$ on G can be written as

$$\varphi(g) = \sum_a \sum_{i,k} C_{ik}^{(a)} \tau_{ik}^{(a)}(g),$$

$$C_{ik}^{(a)} = s_a(\varphi, \tau_{ik}^{(a)})$$

(the last expression follows from the 1st and 2nd orth. theorems).

Proof: We introduce for all $g \in G$ an operator $\hat{R}(g)$:

$$\hat{R}(g)\psi(h) = \psi(hg) \equiv \phi(h).$$

$$\hat{R}(g_1)\hat{R}(g)\psi(h) = \hat{R}(g_1)\psi(hg) = \hat{R}(g_1)\phi(h) =$$

$$= \phi(hg_1) = \psi(hg_1g) = \hat{R}(g_1g)\psi(h)$$

$$\hat{R}(g_1)\hat{R}(g) = \hat{R}(g_1 g)$$

The mapping $g \mapsto \hat{R}(g)$ is the regular representation.
In a general case, R is reducible.

$$\mathcal{L} = \sum_a \mathcal{L}_a$$

Any function of g may be expressed as the linear combination of the basis functions $\varphi_j^{(a)}(g)$ and

$$\varphi_j^{(a)}(g) = \sum_{k=1}^{s_a} \tau_{kj}^{(a)}(g)\varphi_k^{(a)}(g),$$

i.e., any function can be written as $\varphi(g) = \sum_a \sum_{i,k} C_{ik}^{(a)} \tau_{ik}^{(a)}(g)$. The number of the functions in the τ- and φ-basis is the same:

$$\sum_{a=1}^{q} s_a^2 = N$$