Applications of group theory in spectroscopy

Igor Mazets

Atominstitut, TU Wien
http://atomchip.org/theory/lectures/

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What is a group?

A group is a set $\mathcal{G}$ with the following properties:

- A group operation ("multiplication") is defined. It maps each ordered pair of group elements to another group element (called a product):
  $\forall f \in \mathcal{G},\ g \in \mathcal{G}\ \exists$ one and only one $h = fg \in \mathcal{G}$
  (in a general case $fg \neq gf$)
- Associativity:
  $$(g_1 g_2) g_3 = g_1 (g_2 g_3)$$
- Identity element e:
  $\exists e \in \mathcal{G} : \quad \forall f \in \mathcal{G}\ fe = ef = f$
- Inverse element:
  $\forall f \in \mathcal{G}\ \exists f^{-1} \in \mathcal{G} :$
  $$ff^{-1} = f^{-1} f = e$$

Try to prove the uniqueness of e *and* f^{-1}*. Und* $(f^{-1})^{-1} = f$

If the number N of elements of $\mathcal{G}$ is finite, then $\mathcal{G}$ is called finite and N is its order; otherwise $\mathcal{G}$ is called infinite.
If the multiplication is commutative, i.e.,
$\forall f \in \mathcal{G},\ g \in \mathcal{G}\ fg = gf$, then the group is called Abelian.

Examples:

- Real numbers comprise a group w.r.t. addition, zero is the identity element
- Positive real numbers: arithmetic multiplication; 1 is the identity element
- Vectors (translations) in a D-dimensional space w.r.t. addition
- Rotations in a D-dimensional space
- Permutations of n objects (symmetric group S_n)

Which of these groups are Abelian?
Simplest non-trivial example: a group consisting of two elements e and $f = f^{-1}$ (e.g., inversion; S_n)

Conjugate elements

An element g is conjugate to h if $\exists x \in \mathcal{G}: \ xgx^{-1} = h$

- h is also conjugate to g, since $x^{-1}hx = g$
- each element is conjugate to itself
- If g is conjugate to h then g^{-1} is conjugate to h^{-1}
- If g is conjugate to h and h is conjugate to i then g is conjugate to i *(try to prove this)*

All elements of a group that are mutually conjugate comprise a class.
A group is thus partitioned into different classes.
The class containing the identity element consists of e only, since $\forall x \ xex^{-1} = e$.
For Abelian groups, each class consists of a single element only.

Subgroups

If $\mathcal{B} \subseteq \mathcal{G}$ is a group with respect to the same group operation. then $\mathcal{B}$ is a subgroup of $\mathcal{G}$

Examples:

- $\{e\}$ is a trivial subgroup of every group
- Integer numbers (or rational numbers) in a group of real numbers
- Rotations around a given axis in a group of all rotations in 3D
- Permutations that do not involve certain object(s)

Lagrange's theorem: If $\mathcal{G}$ is a finite group of order n and $\mathcal{B}$ is its subgroup of order m, then n/m is integer.

Corollary: If n is a prime number, than $\mathcal{G}$ has no non-trivial subgroups.

Cyclic subgroups

$a \in \mathcal{G}$
Elements e, a^n, $(a^{-1})^n$, where $n = 1, 2, 3, \ldots$ are all natural numbers, comprise a cyclic subgroup.
If $\mathcal{G}$ is finite, there is a finite number of different powers of a and there is the smallest number p such that $a^p = e$. Then the cyclic subgroup is $\{e, a, a^2, \ldots a^{p-1}\}$.

1. *Find examples of cyclic subgroups*
2. *If an order of a finite group is a prime number, what can we say about its cyclic subgroups?*

Homomorphism and isomorphism

Let $\mathcal{G}$ be a group with a group operation $\cdot$
$\mathcal{H}$ be a group with a group operation $*$
φ a function $\mathcal{G} \mapsto \mathcal{H}$.
There is a homomorphism from $\mathcal{G}$ to $\mathcal{H}$ if

$$\forall f \in \mathcal{G},\ g \in \mathcal{G} \quad \varphi(f \cdot g) = \varphi(f) * \varphi(g).$$

If there is one-to-one correspondence between the elements of $\mathcal{G}$ and $\mathcal{H}$, that is, not only $\varphi: \ \mathcal{G} \mapsto \mathcal{H}$ exists, but also $\varphi^{-1}: \ \mathcal{H} \mapsto \mathcal{G}$, then these two groups are isomorphic.
Any proposition, which is true for $\mathcal{G}$, is also true for the isomorphic group (up to the renaming elements and the group operation).

Find examples of isomorphic groups and of homomorphic, but not isomorphic groups.