Examples of groups

Symmetric group

Cycles

Consider a certain permutation (of numbers from 1 to n) $\hat{P}$:

$$\hat{P}k = m_k$$

$$\hat{P} \equiv \begin{pmatrix} 1 & 2 & \dots & n \\ m_1 & m_2 & \dots & m_n \end{pmatrix}$$

Take certain m_0 from 1 to n:

$$m_0,\ m_1 = \hat{P}m_0,\ \dots\ ,\ m_p = \hat{P}m_{p-1}$$

We stop when m_p equals one of the previously used numbers.
Namely, $m_p = m_0$ (*Prove it!*)
$m_0,\ m_1,\ \dots\ ,\ m_{p-1}$ comprise a cycle of length p.

Taking a number not belonging to the already constructed cycle, we can create another cycle for the same $\hat{P}$ and so on. For given $\hat{P}$, the numbers from 1 to n can be organized in cycles in one and only one way. If $Pm_k = m_k$ then this cycle consists of only one number.

Convenient notation: permutation as a product of cycles (single-number cycles may be omitted, for the sake of brevity).

$$\hat{P} \equiv \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \equiv (12)(34)$$

$$\hat{P} \equiv \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \equiv (1342)$$

$$\hat{P} \equiv \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \equiv (24)$$

Conjugate permutations: $\hat{Q}\hat{P}\hat{Q}^{-1}$

$$\hat{P}m_1 = m_2$$

$$\hat{Q}\hat{P}\hat{Q}^{-1}(\hat{Q}m_1) = \hat{Q}\hat{P}m_1 = \hat{Q}m_2$$

If $\hat{P} = (m_1, \ \ldots \ , \ m_j)\ldots(m_l, \ \ldots \ , \ m_{l+i})$ then

$$\hat{Q}\hat{P}\hat{Q}^{-1} = (\hat{Q}m_1, \ \ldots \ , \ \hat{Q}m_j)\ldots(\hat{Q}m_l, \ \ldots \ , \ \hat{Q}m_{l+i})$$

All conjugate permutations have the same structure of cycles.

Classes

All permutations of S_n can be organized in classes with the same structure of cycles.
Partition of an integer n:

$$n = l_1 + l_2 + \cdots + l_j, \qquad l_1 \geq l_2 \geq \cdots \geq l_j$$

Graphically shown as a Young diagram.
# of classes = # number of partitions.

If two permutations belong to the same class, they are conjugate.
If $\hat{P} = (m_1, \ \ldots \ , \ m_j) \ldots (m_l, \ \ldots \ , \ m_{l+i})$,
and $\hat{P}' = (m'_1, \ \ldots \ , \ m'_j) \ldots (m'_l, \ \ldots \ , \ m'_{l+i})$ then

$$\hat{Q} = \begin{pmatrix} m_1 & \ldots & m_j & \ldots & m_l & \ldots & m_{l+i} \\ m'_1 & \ldots & m'_j & \ldots & m'_l & \ldots & m'_{l+i} \end{pmatrix}$$

provides conjugation.

Group $SO(3)$ of rotations of 3D vectors

- $SO(3)$ is non-Abelian
- Rotation is given by 3 parameters:
  (i) 2 angles giving the axis direction + rotation angle
  (ii) 3 Euler angles

Rotations to the same angle $0 \leq \alpha < 2\pi$ around all possible axes comprise a class.

Orthogonal group

$$O(3) = \mathcal{I} \otimes SO(3)$$

Here $\mathcal{I}$ is the inversion group. Inversion commutes with all rotations.