

VIII. Atom in an oscillating magnetic field

Recall that $\vec{e}_{\pm} = \mp \frac{1}{\sqrt{2}} (\vec{e}_x \pm i\vec{e}_y)$ $\vec{e}_0 = \vec{e}_z$

$$A_{\mu} = \vec{A} \cdot \vec{e}_{\mu}$$

$$A_{\pm 1} = \mp \frac{1}{\sqrt{2}} (A_x \pm iA_y) \quad A_0 = A_z$$

$$\vec{A} \cdot \vec{B} = \sum_{\mu=-1}^{+1} A_{\mu} B^{\mu} = \sum_{\mu=-1}^{+1} (-1)^{\mu} A_{\mu} B_{-\mu}$$

d.c. magnetic field $\vec{B}_{d.c.}$ defines the direction of the axis $\hat{z} \Rightarrow$ Zeeman shift (small field)

a.c. magnetic field

$$\vec{B}(t) = B_{0x} \vec{e}_x \cos \omega t + B_{0y} \vec{e}_y \cos(\omega t + \varphi)$$

Arbitrary (in general, elliptic) polarization.
Propagation direction of the e.-m. wave defines the axis \hat{z}' of the rotated system of co-ordinates

Intensity of an e.-m. wave propagating in vacuum

$$I = c \left(\frac{\overline{\epsilon_0 \vec{E}^2}}{2} + \frac{\overline{\vec{B}^2}}{2\mu_0} \right), \quad \text{where } \overline{\dots} - \text{time average}$$

(over a period $2\pi/\omega$)

Both terms in brackets are equal

$$I = \frac{c}{\mu_0} \overline{\vec{B}^2} = \frac{c}{2\mu_0} (B_{0x'}^2 + B_{0y'}^2)$$

Rotationally invariant form

$$I = \frac{c}{2\mu_0} \sum_{i=x,y,z} |B_{0i}|^2 = \frac{2c}{\mu_0} \sum_{\mu=-1}^{+1} |B_{0\mu}|^2$$

where the cyclic components are
introduced via

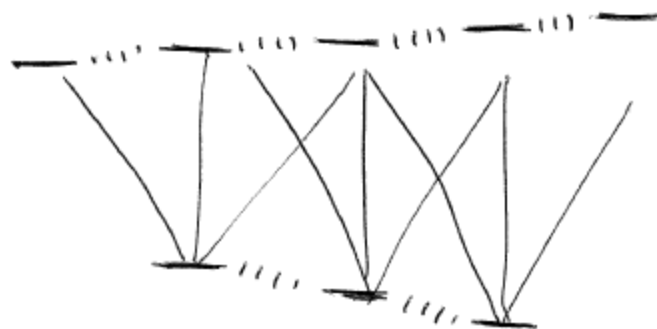
$$\mathbf{B}(t) = (B_{-1} \mathbf{e}^{+1} + B_{+1} \mathbf{e}^{-1}) \exp(-i\omega t) + \text{c.c.}$$

$$B_0 = 0 \quad \text{with respect to the axis } z'$$

We have to find the transition matrix elements
 For ^{87}Rb

$F' = 2$

$F = 1$



— microwave transition (MW)

..... radiofrequency transition (RF)

$$H_{\text{int}} = -\hat{\mu} \vec{B} = -\sum_{\lambda=-1}^{+1} (-1)^{\lambda} \hat{\mu}_{\lambda} B_{-\lambda}$$

Transformation of coordinates:

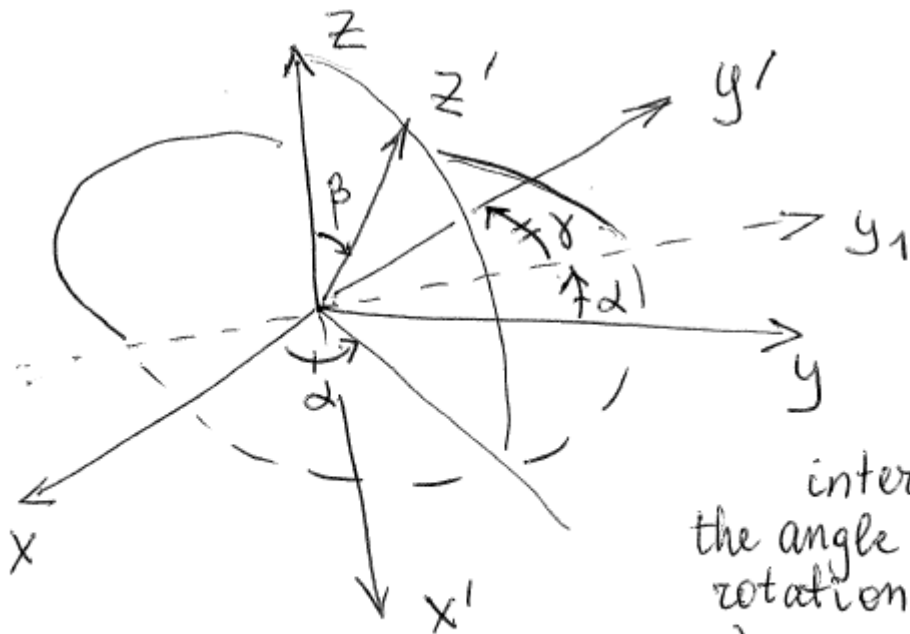
from $S \rightarrow S'$

$$B_{\lambda'} = \sum_{\lambda=-1}^{+1} B_{\lambda} D_{\lambda\lambda'}^1(\alpha, \beta, \gamma)$$

from $S' \rightarrow S$

$$B_{\lambda} = \sum_{\lambda'=-1}^{+1} D_{\lambda\lambda'}^{1*}(\alpha, \beta, \gamma) B_{\lambda'}$$

Recall unitarity relations for $D_{MM'}^J(\alpha, \beta, \gamma)$!



Finding Euler angles:

1) angle between z and $z' = \beta$

2) intersections of the planes (z, z') and

(x, x') defines angle $\alpha =$

Angle between x and this

intersection line, or, equivalently, the angle between the intermediate rotation axis y_1 and (old) y

3) angle between y' and $y_1 = \gamma$

$$B_{\pm 1}'(t) = \mp \frac{1}{\sqrt{2}} \left[B_{0x}' \cos \omega t \pm i B_{0y}' \cos(\omega t + \varphi) \right]$$

$$B_0'(t) = 0$$

$$B_{+1}(t) = e^{i\alpha} \left[d_{11}^1(\beta) e^{i\gamma} B_{+1}'(t) + d_{1-1}^1(\beta) e^{-i\gamma} B_{-1}'(t) \right]$$

$$B_0(t) = \left[d_{01}^1(\beta) e^{i\gamma} B_{+1}'(t) + d_{0-1}^1(\beta) e^{-i\gamma} B_{-1}'(t) \right]$$

$$B_{-1}(t) = e^{-i\alpha} \left[d_{-11}^1(\beta) e^{i\gamma} B_{+1}'(t) + d_{-1-1}^1(\beta) e^{-i\gamma} B_{-1}'(t) \right]$$

□

In the new
coord. system with
z along constant \vec{B}_{dc}

For every $\lambda = -1, 0, +1$ we can write

$$B_\lambda(t) = B_\lambda^{(+)} e^{-i\omega t} + B_\lambda^{(-)} e^{+i\omega t}$$

B_λ is a "non-tilded" IR tensor of the rank 1.

$$(B_\lambda)^* = (-1)^{-\lambda} B_{-\lambda}$$

Hence, $B_\lambda^{(+)} = (-1)^{+\lambda} (B_{-\lambda}^{(-)})^*$.

This can be checked directly, using the properties

$$d_{MM'}^J(\beta) = (-1)^{M-M'} d_{-M-M'}^J(\beta) = (-1)^{M-M'} d_{M'M}^J(\beta) = d_{-M'-M}^J(\beta)$$

$$\hat{H}_{int} = -\hat{\vec{\mu}} \cdot \vec{B} = -\sum_{\lambda=-1}^{+1} (-1)^\lambda \hat{\mu}_\lambda B_{-\lambda} = -\sum_{\lambda=-1}^{+1} (-1)^\lambda \hat{\mu}_{-\lambda} B_\lambda$$

$$\langle JIF' M_F' | \hat{H}_{int} | JIF M_F \rangle =$$

$$= -\sum_{\lambda=-1}^{+1} (-1)^\lambda \langle JIF' M_F' | \hat{\mu}_\lambda | JIF M_F \rangle B_{-\lambda}(t) =$$

$$= -\sum_{\lambda=-1}^{+1} (-1)^\lambda \langle JIF' M_F' | \hat{\mu}_{-\lambda} | JIF M_F \rangle B_\lambda(t)$$

To be definite, consider ^{87}Rb , where the $F = 2$ state is above the $F = 1$ state;

$\hbar\omega_{hfs}$ = hyperfine splitting energy.

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \frac{1}{\hbar} \left\{ \sum_{F'} \sum_{M_F} \left[\hbar\omega_{hfs} \delta_{F'2} + E_{M_F}(F) \right] |F M_F\rangle \langle F M_F| + \hat{H}_{int} \right\} |\Psi\rangle$$

Zeeman energy in a weak constant magnetic field; F and M_F are good quantum numbers.

$$|\Psi\rangle = \sum_F \sum_{M_F} \psi_{FM_F}(t) |JIFM_F\rangle$$

$$i\hbar \frac{\partial}{\partial t} \psi_{FM_F} = (\omega_{hfs} + \hbar^{-1} E_{M_F}(F)) \psi_{FM_F} +$$

$$+ \sum_{F'M_F'} \sum_{\lambda=-1}^{+1} \left(-\frac{1}{\hbar} \right) \langle JIFM_F | \hat{\mu}_\lambda | JIF'M_F' \rangle \cdot (-1)^\lambda \left[B_{-\lambda}^{(+)} e^{-i\omega t} + B_{-\lambda}^{(-)} e^{i\omega t} \right] \psi_{F'M_F'}$$

MICROWAVE TRANSITIONS

$\omega \approx \omega_{hfs}$, the hyperfine components are coupled

$$\psi_{F=1 M_F} \equiv \phi_{F=1 M_F}$$

$$\psi_{F=2 M_F} \equiv \phi_{F=2 M_F} e^{-i\omega t}$$

Then, discarding rapidly [$\sim \exp(\pm i\omega t)$ or $\sim \exp(\pm 2i\omega t)$] terms, we arrive at the system of equations with time-independent coefficients in the r.h.s.

$$i \frac{\partial}{\partial t} \phi_{F=2 M_F} = (\omega_{hfs} - \omega + \hbar^{-1} E_{M_F}(F=2)) \phi_{F=2 M_F} + \sum_{\lambda=-1}^{+1} \frac{(-1)^{\lambda+1}}{\hbar} \langle \mathbf{J} | F=2 M_F | \hat{\mu}_\lambda | \mathbf{J} | F'=1 M_F-\lambda \rangle B_{-\lambda}^{(+)} \phi_{F'=1 M_F-\lambda}$$

$$i \frac{\partial}{\partial t} \phi_{F=1 M_F} = \frac{E_{M_F}(F=1)}{\hbar} \phi_{F=1 M_F} + \sum_{\lambda=-1}^{+1} \frac{(-1)^{\lambda+1}}{\hbar} \langle \mathbf{J} | F=1 M_F | \hat{\mu}_\lambda | \mathbf{J} | F'=2 M_F-\lambda \rangle B_{-\lambda}^{(-)} \phi_{F'=2 M_F-\lambda}$$

This system of equations is Hamiltonian, because

$$B_{\lambda}^{(+)} = (-1)^{\lambda} (B_{-\lambda}^{(-)})^*$$

This set of equations represents the rotating wave approximation (RWA).

RADIOFREQUENCY TRANSITIONS

$$\omega \approx |\Delta E_{\text{Zeeman}} / \hbar|, \text{ where } \Delta E_{\text{Zeeman}} = E_{M_{F+1}}(F) - E_{M_F}(F)$$

is the difference of Zeeman shifts of two adjacent magnetic sublevels.

The hyperfine sublevels with $F=2$ and $F=1$ are off-resonance, their coupling via a.c. magn. field $\sim e^{\pm i\omega t}$ can be neglected.

In general, RF intensity may be high enough to provide $\mu_B B \sim \hbar\omega$. Then the rotating wave approximation does not apply, and we have

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi_{FM_F} &= (\omega_{hfs} \delta_{F2} + \hbar^{-1} E_{M_F}(F)) \psi_{FM_F} + \\ &+ \sum_{M_F \pm 1} + \frac{1}{\hbar} \langle JIF M_F | \hat{M}_{\pm 1} | JIF M_F \mp 1 \rangle (B_{-1}^{(+)} e^{-i\omega t} + B_{-1}^{(-)} e^{i\omega t}) \psi_{FM_F \mp 1} \\ &+ \frac{1}{\hbar} \langle JIF M_F | \hat{M}_{\pm 1} | JIF M_F \pm 1 \rangle (B_{+1}^{(+)} e^{-i\omega t} + B_{+1}^{(-)} e^{i\omega t}) \psi_{FM_F \pm 1} \end{aligned}$$

Write explicitly the equations for RF transitions in the rotating wave approximation.