IV. The Wigner-Eckart theorem

For any irreducible tensor operator

$$\langle n'j'm' | \hat{\mathfrak{M}}_{k\varkappa} | njm \rangle = (-1)^{j'-m'} \begin{pmatrix} j' & k & j \\ -m' & \varkappa & m \end{pmatrix} \langle n'j' \| \hat{\mathfrak{M}}_k \| nj \rangle = (-1)^{2k} C^{j'm'}_{jmk\varkappa} \frac{\langle n'j' \| \hat{\mathfrak{M}}_k \| nj \rangle}{\sqrt{2j'+1}}$$

The dependence on the projections *m*, *m´* , *κ* is reduced to the Clebsch-Gordan coefficient (or, equivalently, to a 3*jm*-symbol).

$\langle n'j' \| \hat{\mathfrak{M}}_k \| nj \rangle$ is called reduced matrix element and does not depend on the orientation of the co-ordinate system.

n, *n´* are all quantum numbers other than *j*,*m* (e.g., radial).

Proof (1st variant)

$$[\hat{J}_{\pm 1}, \mathfrak{M}_{k\kappa}] = \mp\sqrt{\frac{k(k+1)-\kappa(\kappa\pm 1)}{2}}\mathfrak{M}_{k\kappa\pm 1}$$ by definition of an IR tensor

$$\langle n'j'm'|[\hat{J}_{\pm 1}, \mathfrak{M}_{k\kappa}]|njm\rangle = \mp\langle n'j'm'|\sqrt{\frac{k(k+1)-\kappa(\kappa\pm 1)}{2}}\mathfrak{M}_{k\kappa\pm 1}|njm\rangle$$

$$\langle n'j'm'|\hat{J}_{\pm 1} = -(\hat{J}_{\mp 1}|n'j'm'\rangle)^{\dagger}$$

$$\sqrt{k(k+1)-\kappa(\kappa\pm 1)}\langle n'j'm'|\mathfrak{M}_{k\kappa\pm 1}|njm\rangle =$$
$$= \sqrt{j'(j'+1)-m'(m'\mp 1)}\langle n'j'm'\mp 1|\mathfrak{M}_{k\kappa}|njm\rangle -$$
$$- \sqrt{j(j+1)-m(m\pm 1)}\langle n'j'm'|\mathfrak{M}_{k\kappa}|njm\pm 1\rangle$$

$$\Gamma_{\pm}(k,\kappa)\langle n'j'm'|\mathfrak{M}_{k\kappa\pm 1}|njm\rangle + \Gamma_{\pm}(j,m)\langle n'j'm'|\mathfrak{M}_{k\kappa}|njm\pm 1\rangle =$$
$$= \Gamma_{\pm}(j',m'\mp 1)\langle n'j'm'\mp 1|\mathfrak{M}_{k\kappa}|njm\rangle$$

$$\Gamma_{\pm}(j,m) = \sqrt{j(j+1)-m(m\pm 1)}$$

The same recurrence relations that define CG coefficients in Racah‘s method.

And, additionally, for $m + \kappa - 1 = j'$

$$\Gamma_{-}(k,\kappa)C^{j'j'}_{jm\,k\kappa-1} + \Gamma_{-}(j,m)C^{j'j'}_{jm-1\,k\kappa} = 0$$

that follows from

$$C^{cc}_{a\alpha\,b\beta} = \delta_{\alpha+\beta,c}(-1)^{a-\alpha}\left[\frac{(2c+1)!\,(a+b-c)!\,(a+\alpha)!\,(b+\beta)!}{(a+b+c+1)!\,(a-b+c)!\,(-a+b+c)!\,(a-\alpha)!\,(b-\beta)!}\right]^{1/2}$$

$\langle n'j'm'|\mathfrak{M}_{k\kappa}|njm\rangle$ obeys the same set of recurrence relations as $C^{j'm'}_{jm\,k\kappa}$.

Therefore there is a direct proportionality between these values. There is the algebraic proof of the Wigner-Eckart theorem.

Proof (2nd variant)

$$\langle n'j'm'|\mathfrak{M}_{k\kappa}|njm\rangle \equiv \int dX' \,\langle n'j'm'|X'\rangle \mathfrak{M}_{k\kappa}(X')\langle X'|njm\rangle$$

where $\int dX' \ldots$ symbolizes integration over all angular variables and summation over all spin variables, as well as integration over all radial variables.

Now consider the co-ordinates X' as rotated with respect to X by a transformation given by the Euler angles α, β, γ.

The Jacobian of the transformation from X' to X is 1.

$$\langle n'j'm'|\mathfrak{M}_{k\kappa}|njm\rangle =$$

$$= \sum_{m_1',\kappa_1,m_1} \int dX\, \langle n'j'm_1'|X\rangle \mathfrak{M}_{k\kappa_1}(X)\langle X|njm_1\rangle \times$$

$$\times D^{j'\,*}_{m_1'm'}(\alpha,\beta,\gamma) D^{j}_{m_1 m}(\alpha,\beta,\gamma) D^{k}_{\kappa_1\kappa}(\alpha,\beta,\gamma) =$$

$$= \sum_{m_1',\kappa_1,m_1} \langle n'j'm_1'|\mathfrak{M}_{k\kappa_1}|njm_1\rangle D^{j'\,*}_{m_1'm'}(\alpha,\beta,\gamma) D^{j}_{m_1 m}(\alpha,\beta,\gamma) D^{k}_{\kappa_1\kappa}(\alpha,\beta,\gamma)$$

But this rotation can be taken arbitrary. We can average over all possible rotations by integrating and dividing by the integration volume:

$$\langle n'j'm'|\mathfrak{M}_{k\kappa}|njm\rangle = \sum_{m_1',\kappa_1,m_1} \langle n'j'm_1'|\mathfrak{M}_{k\kappa_1}|njm_1\rangle \times$$

$$\times \frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta\, d\beta \int_0^{2\pi} d\gamma\, D^{j'\,*}_{m_1'm'}(\alpha,\beta,\gamma) D^{j}_{m_1 m}(\alpha,\beta,\gamma) D^{k}_{\kappa_1\kappa}(\alpha,\beta,\gamma)$$

Recall that

$$\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma\, D^{J_3 *}_{M_3 M'_3}(\alpha,\beta,\gamma)\, D^{J_2}_{M_2 M'_2}(\alpha,\beta,\gamma)\, D^{J_1}_{M_1 M'_1}(\alpha,\beta,\gamma) = \frac{8\pi^2}{2J_3+1} C^{J_3 M_3}_{J_1 M_1 J_2 M_2} C^{J_3 M_3'}_{J_1 M'_1 J_2 M'_2}$$

$$\langle n'j'm'|\mathfrak{M}_{k\kappa}|njm\rangle = \sum_{m'_1,\kappa_1,m_1} \langle n'j'm'_1|\mathfrak{M}_{k\kappa_1}|njm_1\rangle \frac{1}{2j'+1} C^{j'm'_1}_{jm_1k\kappa_1} C^{j'm'}_{jmk\kappa}$$

Substitution of $\langle n'j'm'|\mathfrak{M}_{k\kappa}|njm\rangle = (-1)^{2k} C^{j'm'}_{jmk\kappa} \frac{\langle n'j'||\mathfrak{M}_{k\kappa}||nj\rangle}{\sqrt{2j'+1}}$

satisfies this equality identically.

Note that $\sum_{\kappa_1,m_1} \left(C^{j'm'_1}_{jm_1k\kappa_1} \right)^2 = 1$

and the subsequent summation over m'_1 removes the prefactor $\frac{1}{2j'+1}$.

Sum rules
$$\sum_{m\varkappa} |\langle n'j'm'|\hat{\mathfrak{M}}_{k\varkappa}|njm\rangle|^2 = \frac{|\langle n'j'\|\hat{\mathfrak{M}}_k\|nj\rangle|^2}{2j'+1},$$

$$\sum_{m'\varkappa} |\langle n'j'm'|\hat{\mathfrak{M}}_{k\varkappa}|njm\rangle|^2 = \frac{|\langle n'j'\|\hat{\mathfrak{M}}_k\|nj\rangle|^2}{2j+1},$$

$$\sum_{mm'} |\langle n'j'm'|\hat{\mathfrak{M}}_{k\varkappa}|njm\rangle|^2 = \frac{|\langle n'j'\|\hat{\mathfrak{M}}_k\|nj\rangle|^2}{2k+1},$$

$$\sum_{mm'\varkappa} |\langle n'j'm'|\hat{\mathfrak{M}}_{k\varkappa}|njm\rangle|^2 = |\langle n'j'\|\hat{\mathfrak{M}}_k\|nj\rangle|^2.$$

follow from the Wigner-Eckart theorem and the normalization of 3*jm*-symbols.

V. Wigner 6j-symbols

Addition of three angular momenta can be realized using three schemes:

$$\begin{aligned}
&\text{I)}\quad \mathbf{j}_1+\mathbf{j}_2=\mathbf{j}_{12}, \quad \mathbf{j}_{12}+\mathbf{j}_3=\mathbf{j},\\
&\text{II)}\quad \mathbf{j}_2+\mathbf{j}_3=\mathbf{j}_{23}, \quad \mathbf{j}_1+\mathbf{j}_{23}=\mathbf{j},\\
&\text{III)}\quad \mathbf{j}_1+\mathbf{j}_3=\mathbf{j}_{13}, \quad \mathbf{j}_{13}+\mathbf{j}_2=\mathbf{j}.
\end{aligned}$$

I) Eigenstates of $\hat{j}_1^2,\ \hat{j}_2^2,\ \hat{j}_3^2,\ \hat{j}_{12}^2,\ \hat{j}^2,\ \hat{j}_z$

$$|j_1 j_2 (j_{12})\, j_3 j m\rangle = \sum_{m_1 m_2 m_3} C^{jm}_{j_{12} m_{12}\, j_3 m_3} C^{j_{12} m_{12}}_{j_1 m_1\, j_2 m_2} |j_1 m_1,\ j_2 m_2,\ j_3 m_3\rangle$$

II) Eigenstates of $\hat{j}_1^2,\ \hat{j}_2^2,\ \hat{j}_3^2,\ \hat{j}_{23}^2,\ \hat{j}^2,\ \hat{j}_z$

$$|j_1,\ j_2 j_3 (j_{23})\, j m\rangle = \sum_{m_1 m_2 m_3} C^{jm}_{j_1 m_1\, j_{23} m_{23}} C^{j_{23} m_{23}}_{j_2 m_2\, j_3 m_3} |j_1 m_1,\ j_2 m_2,\ j_3 m_3\rangle$$

I) Eigenstates of $\hat{j}_1^2,\ \hat{j}_2^2,\ \hat{j}_3^2,\ \hat{j}_{13}^2,\ \hat{j}^2,\ \hat{j}_z$

$$|j_1 j_3 (j_{13})\, j_2 j m\rangle = \sum_{m_1 m_2 m_3} C^{jm}_{j_{13} m_{13}\, j_2 m_2} C^{j_{13} m_{13}}_{j_1 m_1\, j_3 m_3} |j_1 m_1,\ j_2 m_2,\ j_3 m_3\rangle$$

The unitary transformation between the basis functions corresponding to different coupling schemes:

$$\langle j_1 j_2 (j_{12})\, j_3 j m \,|\, j_1, \; j_2 j_3 (j_{23})\, j' m' \rangle = \delta_{jj'} \delta_{mm'} U\,(j_1 j_2 j j_3;\; j_{12} j_{23}) =$$

$$= \delta_{jj'} \delta_{mm'} (-1)^{j_1+j_2+j_3+j} \sqrt{(2j_{12}+1)(2j_{23}+1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}.$$

$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$ – Wigner 6j-symbol

Why the dependence of the transformation coefficients on the total momentum projections is reduced to the Kronecker delta-symbol?

This is due to the Wigner-Eckart theorem:

$$\langle j_1 j_2 (j_{12}) j_3\, jm | j_1, j_2 j_3 (j_{23})\, j'm' \rangle = \langle j_1 j_2 (j_{12}) j_3\, jm | \hat{I}_{00} | j_1, j_2 j_3 (j_{23})\, j'm' \rangle$$

$$\langle j_1 j_2 (j_{12}) j_3\, jm | \hat{I}_{00} | j_1, j_2 j_3 (j_{23})\, j'm' \rangle \propto C^{jm}_{j'm'00} = \delta_{j'j} \delta_{m'm}$$

Identity operator of the rank 0.

It follows from the definition:

$$\sum C^{jm}_{j_{12}m_{12}j_3m_3} C^{j_{12}m_{12}}_{j_1m_1\,j_2m_2} C^{j'm'}_{j_1m_1\,j_{23}m_{23}} C^{j_{23}m_{23}}_{j_2m_2\,j_3m_3} =$$

$$= \delta_{jj'}\delta_{mm'}(-1)^{j_1+j_2+j_3+j}\sqrt{(2j_{12}+1)(2j_{23}+1)}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$$

The summation here is over m_1, m_2, m_3, m_{12}, m_{23}, while the values of m and m' are fixed.

For the two other coupling schemes we have

$$\langle j_1 j_2 (j_{12}) j_3 jm \mid j_1 j_3 (j_{13}) j_2 j'm' \rangle = \delta_{jj'}\delta_{mm'}(-1)^{j+j_1-j_{12}-j_{13}} U(j_2 j_1 j j_3;\ j_{12} j_{13}) =$$

$$= \delta_{jj'}\delta_{mm'}(-1)^{j_2+j_3+j_{12}+j_{13}}\sqrt{(2j_{12}+1)(2j_{13}+1)}\begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{Bmatrix},$$

$$\langle j_1,\ j_2 j_3 (j_{23}) jm \mid j_1 j_3 (j_{13}) j_2 j'm' \rangle = \delta_{jj'}\delta_{mm'}(-1)^{j_2+j_3-j_{23}} U(j_1 j_3 j j_2;\ j_{13} j_{23}) =$$

$$= \delta_{jj'}\delta_{mm'}(-1)^{j_1+j+j_{23}}\sqrt{(2j_{13}+1)(2j_{23}+1)}\begin{Bmatrix} j_1 & j_3 & j_{13} \\ j_2 & j & j_{23} \end{Bmatrix}.$$

All the momenta in $\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$ must be integer or half-integer.

The groups $(j_1 j_2 j_{12})$, $(j_{12} j_3 j)$, $(j_2 j_3 j_{23})$ and $(j_{23} j_1 j)$ must obey the triangle rule.

From the unitarity of the transformation we obtain the orthogonality and normalization conditions:

$$\sum_{j_{12}} (2j_{12}+1)(2j_{23}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j'_{23} \end{Bmatrix} = \delta_{j_{23} j'_{23}},$$

$$\sum_{j_{23}} (2j_{12}+1)(2j_{23}+1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j'_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \delta_{j_{12} j'_{12}}.$$

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{a+b+d+e} W(abed;\ cf)$$

Racah coefficient

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \sum (-1)^{d+e+f+\delta+\varepsilon+\varphi} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} a & e & f \\ \alpha & \varepsilon & -\varphi \end{pmatrix} \begin{pmatrix} d & b & f \\ -\delta & \beta & \varphi \end{pmatrix} \begin{pmatrix} d & e & c \\ \delta & -\varepsilon & \gamma \end{pmatrix}$$

The sum here is formally over $\alpha, \beta, \gamma, \delta, \varepsilon, \varphi,$ but only three summation indices are independent, since the sum of the numbers in the lower row of a 3*jm*-symbol must be zero.

From this expression we can conclude that the value of a 6*j*-symbols remains the same after any transposition of its columns as well as after simultaneous „flipping“ of any two columns (such that the upper and lower elements change their places):

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Bmatrix} a & c & b \\ d & f & e \end{Bmatrix} = \begin{Bmatrix} b & a & c \\ e & d & f \end{Bmatrix} = \begin{Bmatrix} b & c & a \\ e & f & d \end{Bmatrix} = \begin{Bmatrix} c & a & b \\ f & d & e \end{Bmatrix} = \begin{Bmatrix} c & b & a \\ f & e & d \end{Bmatrix} =$$

$$= \begin{Bmatrix} a & e & f \\ d & b & c \end{Bmatrix} = \begin{Bmatrix} a & f & e \\ d & c & b \end{Bmatrix} = \begin{Bmatrix} e & a & f \\ b & d & c \end{Bmatrix} = \begin{Bmatrix} e & f & a \\ b & c & d \end{Bmatrix} = \begin{Bmatrix} f & a & e \\ c & d & b \end{Bmatrix} = \begin{Bmatrix} f & e & a \\ c & b & d \end{Bmatrix} =$$

$$= \begin{Bmatrix} d & e & c \\ a & b & f \end{Bmatrix} = \begin{Bmatrix} d & c & e \\ a & f & b \end{Bmatrix} = \begin{Bmatrix} e & d & c \\ b & a & f \end{Bmatrix} = \begin{Bmatrix} e & c & d \\ b & f & a \end{Bmatrix} = \begin{Bmatrix} c & d & e \\ f & a & b \end{Bmatrix} = \begin{Bmatrix} c & e & d \\ f & b & a \end{Bmatrix} =$$

$$= \begin{Bmatrix} d & b & f \\ a & e & c \end{Bmatrix} = \begin{Bmatrix} d & f & b \\ a & c & e \end{Bmatrix} = \begin{Bmatrix} b & d & f \\ e & a & c \end{Bmatrix} = \begin{Bmatrix} b & f & d \\ e & c & a \end{Bmatrix} = \begin{Bmatrix} f & d & b \\ c & a & e \end{Bmatrix} = \begin{Bmatrix} f & b & d \\ c & e & a \end{Bmatrix}.$$

Regge symmetry properties:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Bmatrix} a & s_1-b & s_1-c \\ d & s_1-e & s_1-f \end{Bmatrix} = \begin{Bmatrix} s_2-a & b & s_2-c \\ s_2-d & e & s_2-f \end{Bmatrix} =$$

$$= \begin{Bmatrix} s_3-a & s_3-b & c \\ s_3-d & s_3-e & f \end{Bmatrix} = \begin{Bmatrix} s_2-d & s_3-e & s_1-f \\ s_2-a & s_3-b & s_1-c \end{Bmatrix} = \begin{Bmatrix} s_3-d & s_1-e & s_2-f \\ s_3-a & s_1-b & s_2-c \end{Bmatrix},$$

where

$$s_1 = \frac{1}{2}(b+c+e+f), \quad s_2 = \frac{1}{2}(a+c+d+f), \quad s_3 = \frac{1}{2}(a+b+d+e).$$

If one of the indices = 0, then

$$\begin{Bmatrix} 0 & b & c \\ d & e & f \end{Bmatrix} = (-1)^{b+e+d} \frac{\delta_{bc}\delta_{ef}}{\sqrt{(2b+1)(2e+1)}}, \quad \begin{Bmatrix} a & b & c \\ 0 & e & f \end{Bmatrix} = (-1)^{a+b+e} \frac{\delta_{bf}\delta_{ce}}{\sqrt{(2b+1)(2c+1)}},$$

$$\begin{Bmatrix} a & 0 & c \\ d & e & f \end{Bmatrix} = (-1)^{a+d+e} \frac{\delta_{ac}\delta_{df}}{\sqrt{(2a+1)(2d+1)}}, \quad \begin{Bmatrix} a & b & c \\ d & 0 & f \end{Bmatrix} = (-1)^{a+b+d} \frac{\delta_{af}\delta_{cd}}{\sqrt{(2a+1)(2c+1)}},$$

$$\begin{Bmatrix} a & b & 0 \\ d & e & f \end{Bmatrix} = (-1)^{a+e+f} \frac{\delta_{ab}\delta_{de}}{\sqrt{(2a+1)(2d+1)}}, \quad \begin{Bmatrix} a & b & c \\ d & e & 0 \end{Bmatrix} = (-1)^{a+b+c} \frac{\delta_{ae}\delta_{bd}}{\sqrt{(2a+1)(2b+1)}}.$$

$$\begin{Bmatrix} a & b & c \\ 1/2 & e & f \end{Bmatrix}$$

f	$e = c + 1/2$	$e = c - 1/2$
$b + 1/2$	$(-1)^{s+1} \frac{1}{2} \left[\frac{(s+2)(s-2a+1)}{(2b+1)(b+1)(2c+1)(c+1)} \right]^{1/2}$	$(-1)^{s} \frac{1}{2} \left[\frac{(s-2c+1)(s-2b)}{(2b+1)(b+1)c(2c+1)} \right]^{1/2}$
$b - 1/2$	$(-1)^{s} \frac{1}{2} \left[\frac{(s-2c)(s-2b+1)}{b(2b+1)(2c+1)(c+1)} \right]^{1/2}$	$(-1)^{s} \frac{1}{2} \left[\frac{(s+1)(s-2a)}{b(2b+1)c(2c+1)} \right]^{1/2}$

$$s = a + b + c,$$

$$\begin{Bmatrix} a & b & c \\ 1 & e & f \end{Bmatrix}$$

f	$e = c + 1$
$b + 1$	$(-1)^{s} \frac{1}{2} \left[\frac{(s+2)(s+3)(s-2a+1)(s-2a+2)}{(2b+1)(b+1)(2b+3)(2c+1)(c+1)(2c+3)} \right]^{1/2}$
b	$(-1)^{s+1} \frac{1}{2} \left[\frac{(s+2)(s-2c)(s-2b+1)(s-2a+1)}{b(2b+1)(b+1)(2c+1)(c+1)(2c+3)} \right]^{1/2}$
$b - 1$	$(-1)^{s} \frac{1}{2} \left[\frac{(s-2c-1)(s-2c)(s-2b+1)(s-2b+2)}{(2b-1)b(2b+1)(2c+1)(c+1)(2c+3)} \right]^{1/2}$

$$\begin{Bmatrix} a & b & c \\ 1 & e & f \end{Bmatrix}$$ (continued)

f	$e = c$
$b+1$	$(-1)^{s+1}\frac{1}{2}\left[\frac{(s+2)(s-2c+1)(s-2b)(s-2a+1)}{(2b+1)(b+1)(2b+3)c(2c+1)(c+1)}\right]^{1/2}$
b	$(-1)^{s+1}\frac{1}{2}\frac{X}{[b(2b+1)(b+1)c(2c+1)(c+1)]^{1/2}}$
$b-1$	$(-1)^{s}\frac{1}{2}\left[\frac{(s+1)(s-2c)(s-2b+1)(s-2a)}{(2b-1)b(2b+1)c(2c+1)(c+1)}\right]^{1/2}$

f	$e = c - 1$
$b+1$	$(-1)^{s}\frac{1}{2}\left[\frac{(s-2c+1)(s-2c+2)(s-2b-1)(s-2b)}{(2b+1)(b+1)(2b+3)(2c-1)c(2c+1)}\right]^{1/2}$
b	$(-1)^{s}\frac{1}{2}\left[\frac{(s+1)(s-2c+1)(s-2b)(s-2a)}{b(2b+1)(b+1)(2c-1)c(2c+1)}\right]^{1/2}$
$b-1$	$(-1)^{s}\frac{1}{2}\left[\frac{s(s+1)(s-2a-1)(s-2a)}{(2b-1)b(2b+1)(2c-1)c(2c+1)}\right]^{1/2}$

$$X = -a(a+1) + b(b+1) + c(c+1)$$