XIII. Schwinger model for the angular momentum operator

Consider two independent bosonic modes described by annihilation/creation operator obeying the standard commutation relation:

$$[\hat{a},\hat{a}^{+}]=1, \quad [\hat{b},\hat{b}^{+}]=1, \quad [\hat{a},\hat{b}]=0, \quad [\hat{a},\hat{b}^{+}]=0$$

Then one can show that the operators

$$\hat{J}_x = \frac{\hat{a}^{+}\hat{b}+\hat{b}^{+}\hat{a}}{2}, \quad \hat{J}_y = \frac{\hat{a}^{+}\hat{b}-\hat{b}^{+}\hat{a}}{2i}, \quad \hat{J}_z = \frac{\hat{a}^{+}\hat{a}-\hat{b}^{+}\hat{b}}{2}$$

satisfy the commutation relation for the components of the angular momentum operator

$$[\hat{J}_x,\hat{J}_y]=i\hat{J}_z, \quad [\hat{J}_y,\hat{J}_z]=i\hat{J}_x, \quad [\hat{J}_z,\hat{J}_x]=i\hat{J}_y.$$

Also $[\hat{J}_\ell,\hat{J}^2]=0, \ \ell = x,y,z,$ where

$$\hat{J}^2 \equiv \hat{J}_x^2+\hat{J}_y^2+\hat{J}_z^2=\left(\frac{\hat{N}}{2}+1\right)\frac{\hat{N}}{2}, \qquad \hat{N}\equiv \hat{a}^{+}\hat{a}+\hat{b}^{+}\hat{b}$$

Eigenvalues N of $\hat{N}$ are non-negative integers ➡ the momentum $J = N/2$ is half-integer ≥ 0.

Cyclic components $\hat{J}_{\pm1} = \mp\frac{1}{\sqrt{2}}\left(\hat{J}_x \pm i\,\hat{J}_y\right), \quad \hat{J}_0 = \hat{J}_z$

$\hat{J}_{+1} = -\frac{\hat{a}^+\hat{b}}{\sqrt{2}}$ raises M by 1,

$\hat{J}_{-1} = \frac{\hat{b}^+\hat{a}}{\sqrt{2}}$ lowers M by 1, where M is an eigenvalue of $\hat{J}_0$.

Holstein–Primakoff transformation

Mapping of quantum ang.momentum to bosonic annihilation/creation operators. Consider $|\,J, M = +J>$ as a vacuum state and, respectively, $m = J - M$ as the number of excitations. Introduce formally bosonic operators $\hat{c},\ \hat{c}^\dagger,\ [\hat{c},\hat{c}^\dagger] = 1$

$$|J, M = J - m\rangle = (m!)^{-1/2} c^{\dagger\, m} |\mathrm{vac}\rangle$$

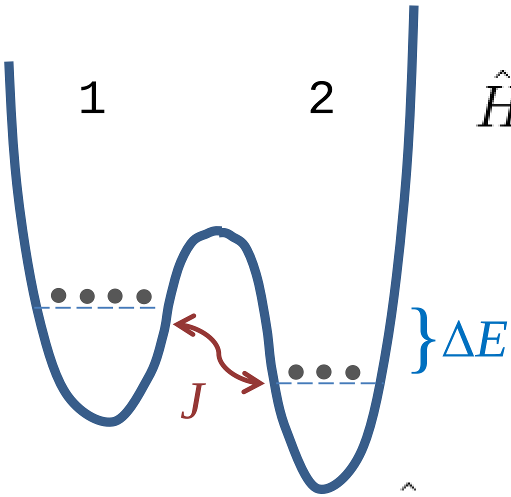
Then, recalling the expression for the matrix elements of cyclic components of $\hat{\mathbf{J}}$ we obtain

$$\hat{J}_0 = J - \hat{c}^\dagger\hat{c}\,, \quad \hat{J}_{+1} = -\sqrt{J}\sqrt{1 - \frac{\hat{c}^\dagger\hat{c}}{2J}}\,\hat{c}, \quad \hat{J}_{-1} = -\sqrt{J}\,\hat{c}^\dagger\sqrt{1 - \frac{\hat{c}^\dagger\hat{c}}{2J}}$$

This transformation is especially convenient for the small number of excitations, $m << J$, where one can expand these expressions in Taylor series in $\frac{\hat{c}^\dagger\hat{c}}{2J}$.

XIV. Quantum models to be mapped on angular-momentum problems

(XIV.1) Two-mode Bose-Hubbard model (ultracold atoms in a double well potential – the simplest, 2-mode description)



$$\hat{H} = U_1 \hat{N}_1^2 + U_2 \hat{N}_2^2 + \frac{\Delta E}{2}(\hat{N}_1 - \hat{N}_2) - J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)$$

$$\hat{N}_j = \hat{a}_j^\dagger \hat{a}_j, \qquad j = 1, 2$$

$$\hat{N} = \hat{N}_1 + \hat{N}_2, \qquad [\hat{N}, \hat{H}] = 0$$

$$\hat{S}_z = \frac{\hat{N}_1 - \hat{N}_2}{2}, \qquad \hat{S}_x = \frac{\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1}{2}, \qquad \hat{S}^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right)$$

$$\hat{H} = (U_1 + U_2)\frac{N^2}{4} + (U_1 + U_2)\hat{S}_z^2 + [\Delta E + N(U_1 - U_2)]\hat{S}_z - 2J\hat{S}_x$$

(XIV.2) Dicke model

A two-level system consisting of two states, ground |g> and excited |e>, is formally equivalent to a (pseudo)spin s = 1/2.

The raising operator

$$\hat{\sigma}^{+} = \hat{s}_x + i\,\hat{s}_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

transformes |g> into |e>,

the lowering operator

$$\hat{\sigma}^{-} = \hat{s}_x - i\,\hat{s}_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

transforms |e> into |g>.

The operator of the population difference

$$\hat{\sigma}_z = 2\,\hat{s}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hamiltonian of N two-level system with a single electromagnetic mode (practically, with a cavity mode). We denote the photon annihilation operator by $\hat{a}$.
If we assume that the atom-photon coupling constant is the same for all atoms, this Hamiltonian reads as (we set Planck´s constant $\hbar = 1$)

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \sum_{i=1}^{N} [\omega_0 \hat{s}_{z\,i} + 2g(\hat{a}^\dagger + \hat{a})\hat{s}_{x\,i}]$$

Sum of individual spin operators yield the collective spin operator:

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{S}_z + 2g(\hat{a}^\dagger + \hat{a})\hat{S}_x$$

Since the e.m.-mode is close to the resonance, $\omega \approx \omega_0$, we can use the rotating wave approximation (RWA):

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{S}_z + g(\hat{a}^\dagger \hat{S}^- + \hat{a}\,\hat{S}^+)$$

What is the integral of motion of this Hamiltonian?

Note: the same coupling constant for all atoms may be attained for a ring (running-wave) cavity; the phase factors exp (i**k**$\mathbf{r}_j$) for different atoms can be included into the definition of |e>.
The use of the Holstein-Primakoff transformation reduces the Hamiltonian to one for two bosonic fields (atomic excitations and phonons). How this bosonic Hamiltonian looks if the number of at.excitations + the number of photons << N? In the case of small number of excitations and phonons write the Hamiltonian in the case of non-equal coupling constant (each atom possessing its own g_j).

But N pseudospins $s = 1/\,2$ may be summed in different ways.
If they form a fully simmetrized state, i.e., characterized by the Young diagram $\{N\}$, then we obtain max.possible collective spin $S = N/2$.
In a general case, for the Young diagram $\{N - m, m\}$, where $m \leq N/2$, we obtain
$S = N/2 - m$.
In particular, for an even N and $m = N/2$ (the Young diagram consisting of two rows of the equal length) $S = 0$.

The rate Γ of photon emission into the cavity mode is proportional to $\langle \hat{S}^+ \hat{S}^- \rangle$

If (almost) all atoms are in the |e> state, $\langle \hat{S}_z \rangle \approx S$, then $\Gamma \propto S$.
When in the course of evolution, almost half of the atoms decayed into the state |g>, i.e.,

when $\langle \hat{S}_z \rangle \approx 0$, we obtain $\Gamma \propto S^2$.
The states with $\{\lambda\} = \{N\}$ and, hence, $S = N/2$ are called Dicke states. They are charcterized by the maximum possible photon emission rate

$\langle \hat{S}_z \rangle \approx N/2$ ➡ $\Gamma \propto N$ Atoms emit photons independently.

$\langle \hat{S}_z \rangle \approx 0$ ➡ $\Gamma \propto N^2$ Collective (enhanced) emission – superradiance.

The opposite limit: states with $\{\lambda\} = \{N/2, N/2\}$ for even N and, hence, $S = 0$, do not emit into the cavity mode at all. Do they emit into other modes (side modes)? Why?

Single-electron qubits: besides the pseuspoin, there is the spin of electron
Wave function of N electrons:

$$|\Psi\rangle = \hat{\mathcal{A}}|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle$$

antisymmetrization

$$|\psi_j\rangle = |pseudospin_j\rangle \otimes |location_j\rangle \otimes |spin_j\rangle$$

$\{\lambda_{\text{loc}}\}$ $\{\lambda_{\text{spin}}\}$

IR tensor product (symmetric group S_N)

$\{\lambda_{\text{pseudospin}}\}$ $\{\tilde{\lambda}_{\text{pseudospin}}\}$

IR tensor product (symmetric group S_N)

$\{1, 1, 1, \ldots, 1\} \equiv \{1^N\}$

(XIV.3) Calculation of matrix elements of the contact interaction in the basis of harmonic oscillator wave functions

$$U(\vec{r}_1 - \vec{r}_2) = g\,\delta(\vec{r}_1 - \vec{r}_2)$$

$$\mathcal{M}_{\vec{n}_1' \vec{n}_2', \vec{n}_1 \vec{n}_2} = \int d^3\vec{r}_1 \int d^3\vec{r}_2\, \psi_{\vec{n}_1'}(\vec{r}_1)\, \psi_{\vec{n}_2'}(\vec{r}_2) \times$$

$$\times g\,\delta(\vec{r}_1 - \vec{r}_2)\, \psi_{\vec{n}_2}(\vec{r}_2)\, \psi_{\vec{n}_1}(\vec{r}_1)$$

$$\vec{n} = \{n_x, n_y, n_z\}$$

$$\psi_{\vec{n}}(\vec{r}) = \psi_{n_x}(x)\, \psi_{n_y}(y)\, \psi_{n_z}(z)$$

$$\bar{x} = \frac{x}{l_x}, \quad l_x = \sqrt{\frac{\hbar}{m\omega_x}}$$ etc.

$$\mathcal{M}_{\vec{n}_1'\vec{n}_2', \vec{n}_1\vec{n}_2} = \frac{g}{l_x l_y l_z} \prod_{i=x}^{z} \bar{\mathcal{M}}_{n_{i1}' n_{i2}' n_{i1} n_{i2}}$$

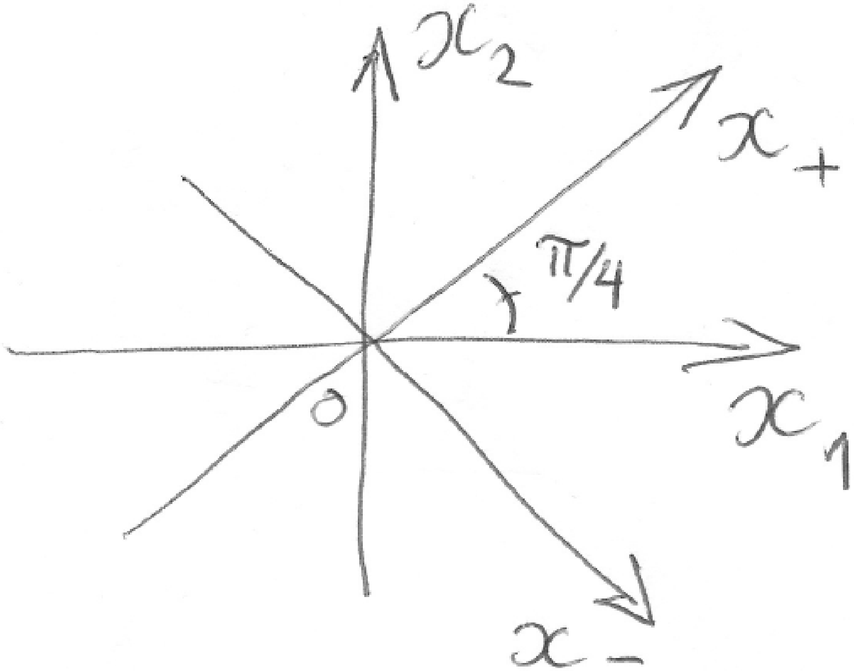
$$\mathcal{M}_{n_1' n_2' n_1 n_2} = \int dx_1 \int dx_2 \, \psi_{n_1'}(x_1) \psi_{n_2'}(x_2) \cdot$$

$$\times \delta(x_1 - x_2) \psi_{n_2}(x_2) \psi_{n_1}(x_1)$$

$$\psi_{n_1}(x_1) = \frac{\hat{a}_1^{+\,n_1}}{\sqrt{n_1!}} |0\rangle, \quad \hat{a}_1^+ = \frac{x_1 - \frac{\partial}{\partial x_1}}{\sqrt{2}}$$

Schwinger model: $$\hat{J}_z = \frac{1}{2}(\hat{a}_1^+ \hat{a}_1 - \hat{a}_2^+ \hat{a}_2)$$

$$\psi_{n_1}(x_1)\psi_{n_2}(x_2) = \left| J = \frac{n_1 + n_2}{2}, M = \frac{n_1 - n_2}{2} \right\rangle$$



$$x_{\pm} = \frac{x_1 \pm x_2}{\sqrt{2}}$$

$$\delta(x_1 - x_2) = \delta(\sqrt{2}\, x_-) = \frac{1}{\sqrt{2}}\,\delta(x_-)$$

x_1, x_2 – „new“ co-ordinates, x_-, x_+ – „old“ co-ordinates

$$\Psi_{JM}(\Omega^{new}) = \sum_{M'=-J}^{J} \Psi_{JM'}(\Omega^{old})\, \mathcal{D}^{J}_{M'M}(\alpha,\beta,\gamma)$$

In our case $\alpha = \gamma = 0$, $\beta = \pi/2$

$$M' = \frac{n_- - n_+}{2}$$

$$D^{J}_{MM'}\left(0, \frac{\pi}{4}, 0\right) \equiv d^{J}_{MM'}\left(\frac{\pi}{4}\right)$$

$$n_- + n_+ = n_1 + n_2 \equiv 2J$$

$$\psi_{n_1}(x_1)\psi_{n_2}(x_2) = \sum_{n_+=0}^{n_1+n_2} \psi_{n_1+n_2-n_+}(x_-)\psi_{n_+}(x_+) D^{\frac{1}{2}(n_1+n_2)}_{\frac{1}{2}(n_1+n_2)-n_+ \ \frac{1}{2}(n_1-n_2)}(0, \tfrac{\pi}{4}, 0)$$

$$\int_{-\infty}^{\infty} dx_+ \, \psi_{n_+}(x_+)\psi_{n'_+}(x_+) = \delta_{n_+ n'_+}$$

$$\bar{\mathcal{M}}_{n'_1\, n'_2,\, n_1\, n_2} = \frac{1}{\sqrt{2}} \sum_{n_+=0}^{n_{+\,\max}} \psi_{n_1+n_2-n_+}(0)\psi_{n'_1+n'_2-n_+}(0) \times$$

$$\times D^{\frac{1}{2}(n_1+n_2)}_{\frac{1}{2}(n_1+n_2)-n_+ \ \frac{1}{2}(n_1-n_2)}(0, \tfrac{\pi}{4}, 0) D^{\frac{1}{2}(n'_1+n'_2)}_{\frac{1}{2}(n'_1+n'_2)-n_+ \ \frac{1}{2}(n'_1-n'_2)}(0, \tfrac{\pi}{4}, 0)$$

$$n_{+\,\max} = \min(n_1 + n_2,\, n'_1 + n'_2)$$

In the dimensionless form:

$$\psi_n(0) = \begin{cases} \dfrac{(-1)^{n/2}(n-1)!!}{\sqrt{\sqrt{\pi}n!}}, & n = 0, 2, 4, 6, \ldots \\ 0, & n = 1, 3, 5, 7, \ldots \end{cases}$$

$D_{M}{}^{J}{}_{M'}(0,\beta,0) \equiv d_{M}{}^{J}{}_{M'}(\beta)$ can be expressed via

- Jacobi polynomials

$$d^J_{MM'}(\beta) = \xi_{MM'}\left[\frac{s!\,(s+\mu+\nu)!}{(s+\mu)!\,(s+\nu)!}\right]^{1/2}\left(\sin\frac{\beta}{2}\right)^{\mu}\left(\cos\frac{\beta}{2}\right)^{\nu} P_s^{(\mu,\nu)}(\cos\beta)$$

- hypergeometric function

$$d^J_{MM'}(\beta) = \frac{\xi_{MM'}}{\mu!}\left[\frac{(s+\mu+\nu)!\,(s+\mu)!}{s!\,(s+\nu)!}\right]^{1/2}\left(\sin\frac{\beta}{2}\right)^{\mu}\left(\cos\frac{\beta}{2}\right)^{\nu}\times$$

$$\times F\left(-s,\ s+\mu+\nu+1;\ \mu+1;\ \sin^2\frac{\beta}{2}\right)$$

$$\mu = |M-M'|,\ \nu = |M+M'|,\ s = J-\frac{1}{2}(\mu+\nu)$$

$$\xi_{MM'} = \begin{cases} 1 & \text{при } M' \geqslant M, \\ (-1)^{M'-M} & \text{при } M' < M \end{cases}$$