XII. Addition of many identical spins

(XII.1). Symmetric group

Symmetric group $S_n$ is the group of all possible permutations of $n$ objects. In total $n!$ elements (including identity operation). Each permutation is a product of a certain finite number of *pairwise* transpositions. For a given permutation, this number is always even or always odd (the same permutation can be achieved by different sequencies of transpositions). Therefore, we can speak about even and odd permutations. The sign of a permutation $P$:

$$\text{sgn}(P) = \begin{cases} +1 & \text{, } P \text{ even} \\ -1 & \text{, } P \text{ odd} \end{cases}$$
Irreducible representations of the symmetric group

IRs of $S_n$ correspond to conjugacy classes of $S_n$ and are labelled by a partition of the integer, positive number $n$.

$$\{\lambda\} \equiv \{\lambda_1, \lambda_2, \ldots, \lambda_r\}, \quad n = \lambda_1 + \lambda_2 + \ldots + \lambda_r.$$ 

We order these integer number as

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0.$$ 

This partition can be represented graphically by so-called Young diagrams. A Young diagram contains $\lambda_j$ boxes in its $j$th row.

Few examples:

\begin{align*}
\{2\} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=2cm]{young1}}
\end{array}
\end{array}
\end{array} \\
\{1,1\} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=2cm]{young2}}
\end{array}
\end{array}
\end{array} \\
\{3,3,2,1,1\} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=2cm]{young3}}
\end{array}
\end{array}
\end{array}
\end{align*}
Conjugate diagrams: rows ↔ columns, or, equivalently, reflection with respect to the diagonal.

\[ \lambda = \{3,3,2,1,1\} \quad \lambda = \{5,3,2\} \]

Conjugation is denoted by \( \sim \).

Young tableau: a Young diagram (n boxes) filled with integer numbers 1,2, ... , n.
Standard Young tableau: numbers in each row and in each column are placed in the increasing order.
Example: all standard Young tableaux for \( \lambda = \{3,2\} \).

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & \\
\end{array} \quad \begin{array}{cccc}
1 & 2 & 4 \\
3 & 5 & \\
\end{array} \quad \begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & \\
\end{array} \quad \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & \\
\end{array} \quad \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 & \\
\end{array}
\]
The dimension $d_{\{\lambda\}}$ of an IR of $S_n$ characterized by the Young diagram $\{\lambda\}$ is equal to the number of corresponding standard Young tableaux. The dimensions of IRs characterized by conjugate Young diagrams are the same,
\[
d_{\{\lambda\}} = d_{\{\bar{\lambda}\}}
\]
Note that
\[
\sum_{\{\lambda\}} d_{\{\lambda\}}^2 = n!
\]

Hook length $h(j)$ of the $j$th box of a Young diagram:

\[
h(j) = \text{number of boxes to the right in the row} + \text{number of boxes below in the column} + 1
\]

Hook length for some $\{\lambda\}$

An easier way to calculate $d_{\{\lambda\}}$ is given by a theorem:
\[
d_{\{\lambda\}} = \frac{n!}{\prod_{j=1}^{n} h(j)}
\]
(XII.2). Systems of identical particles.

Consider \( n \) identical particles with the spin \( s \).

Simultaneous permutation of both co-ordinates \( r_j \) and spin variables \( \sigma_j \) of each pair of particles multiplies the \( n \)-particle wave function by +1 if \( s \) is integer (bosons, B) and by –1 if \( s \) is half-integer (fermions, F).

A wave function satisfying this symmetry requirement can be constructed in many ways, corresponding to various Young diagrams

\[
\Psi_B(r_1, \ldots, r_n; \sigma_1, \ldots, \sigma_n) = \sum_\tau \Phi_{\{\lambda\}_\tau}(r_1, \ldots, r_n)X_{\{\lambda\}_\tau}(\sigma_1, \ldots, \sigma_n)
\]

\[
\Psi_F(r_1, \ldots, r_n; \sigma_1, \ldots, \sigma_n) = \sum_\tau \Phi_{\{\tilde{\lambda}\}_\tau}(r_1, \ldots, r_n)X_{\{\lambda\}_\tau}(\sigma_1, \ldots, \sigma_n)
\]

Here sum is taken over all \( d_{\{\lambda\}} \) standard Young tableaux, where we use spin variables \( \sigma_j \) instead of numbers \( j = 1, 2, \ldots, n \) to fill the boxes, when we construct the spin part \( X \). The co-ordinate part is obtained by replacing the spin variables with the co-ordinates and, in the case of fermions only, by conjugating the Young tableau.

We focus on the construction of the spin part \( X \) of the wave function, the co-ordinate part being built in a similar way.
1. Choose a certain standard Young tableau, like that:

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\[ \sigma_1 \sigma_3 \sigma_6 \\
\sigma_2 \sigma_4 \\
\sigma_5 \sigma_7 \]
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2. Construct a product of single spin wave functions

\[ \chi(\sigma_1, \ldots, \sigma_n) = \prod_{j=1}^{n} \langle \sigma_j | s m_s^j \rangle \]

where \( m_s^j = -s, -s+1, \ldots, s-1, s \) is the projection of the spin of the \( j \)th particle to the quantization axis.

3. Apply to this product a **Young symmetrizer**.

For a given Young tableau we select from all \( n! \) permutations only those operations (including the identity operation!), which do not permute spin variables belonging to different rows. We denote these permutations by \( P_r \).

**How many are there \( P_r \) ’s?**

Also we select permutations (including identity!), which do not permute spin variables belonging to different columns, denoting them by \( P_c \).

**How many are there \( P_c \) ’s?**

Then, summing by all possible \( P_r \) and \( P_c \), we define the Young symmetrizer:

\[ Y_{\{\lambda\},\tau} = \sum_{P_r, P_c} sgn(P_c) P_r P_c \]
Spin wave function

\[ X_{\{\lambda\} \tau} (\sigma_1, \ldots, \sigma_n) = Y_{\{\lambda\} \tau} \chi (\sigma_1, \ldots, \sigma_n) \]

The maximum number of row in \{\lambda\} is \(2s + 1\), since antisymmetrization over variables in columns containing more than \(2s + 1\) will require antisymmetrization over arguments \(\sigma_j\) and \(\sigma_i\) of two functions \(<\sigma_j|s m_s>\) and \(<\sigma_i|s m_s>\) with the same \(m_s\). The result of the antisymmetrization will then be identically zero.

In particular, for spin-1/2 particles, such as electrons, the spin wave function can be characterized by Young diagrams with 1 row (fully symmetric) or 2 rows only.

It is possible to calculate, which values of the total spin \(S\) built from \(n\) identical spins \(s\) correspond to a given Young diagram.

If the total spin \(S\) appears \(W_{S\{\lambda\}}\) times for the \{\lambda\}-type wave function than, obviously

\[
\sum_{\{\lambda\}} \sum_{S} W_{\{\lambda\} S} (2S + 1) = (2s + 1)^n
\]
For $s = \frac{1}{2}$, unlike other (non-zero) values of particle spin, there is a one-to-one correspondence between $\{\lambda\}$ and $S$.

The rule:
If we add to a state of $n$ spin-$\frac{1}{2}$ particles with the total spin $S$ and $\{\lambda\} = \{\lambda_1, \lambda_2\}$, $\lambda_1 + \lambda_2 = n$, a new particle of the same kind, then we obtain the states:

1) $\{\lambda_1 + 1, \lambda_2\}$, $S + \frac{1}{2}$,

2) $\{\lambda_1, \lambda_2 + 1\}$, $S - \frac{1}{2}$. The variant (2) is possible for $\lambda_1 > \lambda_2$ only.

Recall that $W$ shows how many times the total spin $S$ appears for a given $\{\lambda\}$. 
Addition of spins $s = 1/2$.

Young diagrams not appearing for $s = 1/2$.

For $s > 1/2$, for a given $\{\lambda\}$, $W$ may be non-zero for different $S$.

...and so on.
Explicit construction of wave functions of many spins $s = \frac{1}{2}$

For each particle two states: ,,spin up“ and ,,spin-down“.

Spin-lowering operator for $j$th particle:

$$\sigma_j^- |\uparrow\rangle_j = |\downarrow\rangle_j, \quad \sigma_j^- |\downarrow\rangle_j = 0$$

$$\sigma_j^- |\uparrow\rangle_i = 0, \quad \sigma_j^- |\downarrow\rangle_i = 0 \quad \text{for} \ i \neq j$$

Collective spin-lowering operator:

$$\sigma^- = \sum_{j=1}^{n} \sigma_j^-$$

The highest possible $S = n/2$ corresponds to the fully symmetrized state (Young diagram $\{n\}$ consists of a single row).

The highest possible collective spin projection $M = +S = +n/2$: all atoms in the spin-up state.

$$|\{n\}, S = n/2, M = +n/2\rangle = \prod_{j=1}^{n} |\uparrow\rangle_j$$

Applying the collective spin-lowering operator, we decrease $M$ by 1:

$$|\{n\}, S = n/2, M = (n/2) - 1\rangle = \frac{1}{\sqrt{n!}} \sigma^- |\{n\}, S = n/2, M = n/2\rangle =$$

$$= \frac{1}{\sqrt{n!}} \sum_{i=1}^{n} |\downarrow\rangle_i \prod_{j \neq i} |\uparrow\rangle_j$$
But there are \( n \) linearly independent states with \( M = (n/2) - 1 \): one spin down, all other spins up:
\[
\left| \downarrow_i \prod_{j \neq i} \downarrow_j \right>, \quad i = 1, 2, \ldots, n.
\]
From them we can construct \((n - 1)\) linearly independent functions orthogonal to
\[
\left| \{n\}, S = n/2, M = +n/2 \right> = \prod_{j=1}^{n} \left| \uparrow_j \right>.
\]
These \((n - 1)\) functions are to be identified as different realizations of \( S = (n/2) - 1 \) for the Young diagram \( \{n - 1, 1\} \).

We apply \( \sigma^- \) once again and obtain \( n \) functions for the states with \( S = n/2 \) and \( S = (n/2) - 1 \) with the total spin projection \( M = (n/2) - 2 \).

But there are \( n(n - 1)/2 \) linearly independent states with 2 spins down and \((n - 1)\) spins up. By orthogonalizing them to the already defined states, we obtain
\[
n(n - 1)/2 - n = n(n - 3)/2 \text{ states for } S = (n/2) - 2 \text{ and } \{\lambda\} = \{n - 2, 2\}.
\]

\( \{\lambda\} = \{n - m, m\} \), where \( m \) is integer and \( m \leq n/2 \) corresponds to \( S = (n/2) - m \).

Calculate \( W_{\{\lambda\}, S} \) for \( \{\lambda\} = \{n - m, m\} \), \( S = (n/2) - m \).