II. Addition of angular momenta

0´. Reminder: properties of the angular momentum operator

Hermiticity: $\hat{\mathbf{J}}^+ = \hat{\mathbf{J}}$

Commutation relations for Cartesian components

$$[\hat{J}_i,\ \hat{J}_k] = i\varepsilon_{ikl}\hat{J}_l,\ [\hat{\mathbf{J}}^2,\ \hat{J}_i] = 0\ \ (i,\ k,\ l = x,\ y,\ z)$$

$$\hat{\mathbf{J}}^2 = \sum_i \hat{J}_i^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

The same for covariant cyclic components

$$[\hat{J}_{+1},\ \hat{J}_{+1}] = [\hat{J}_0,\ \hat{J}_0] = [\hat{J}_{-1},\ \hat{J}_{-1}] = 0,$$

$$[\hat{J}_{+1},\ \hat{J}_0] = -[\hat{J}_0,\ \hat{J}_{+1}] = -\hat{J}_{+1},\ [\hat{J}_{+1},\ \hat{J}_{-1}] = -[\hat{J}_{-1},\ \hat{J}_{+1}] = -\hat{J}_0,$$

$$[\hat{J}_0,\ \hat{J}_{-1}] = -[\hat{J}_{-1},\ \hat{J}_0] = -\hat{J}_{-1},$$

$$[\hat{\mathbf{J}}^2,\ \hat{J}_{+1}] = [\hat{\mathbf{J}}^2,\ \hat{J}_0] = [\hat{\mathbf{J}}^2,\ \hat{J}_{-1}] = 0.$$

$$\hat{\mathbf{J}}^2 = \sum_\mu (-1)^\mu \hat{J}_{-\mu}\hat{J}_\mu = -\hat{J}_{+1}\hat{J}_{-1} + \hat{J}_0\hat{J}_0 - \hat{J}_{-1}\hat{J}_{+1} = \hat{J}_0^2 - \hat{J}_0 - 2\hat{J}_{+1}\hat{J}_{-1} = \hat{J}_0^2 + \hat{J}_0 - 2\hat{J}_{-1}\hat{J}_{+1}$$

$$(\hat{J}_\mu)^+ = \hat{J}^\mu = (-1)^\mu\,\hat{J}_{-\mu},\ \ (\mu = \pm 1,\ 0)$$

The angular momentum is a pseudovector, i.e., it does not change sign after the inversion of the co-ordinates (**r** → – **r**):

$$\hat{P}_r \hat{J}_i \hat{P}_r^{-1} = \hat{J}_i, \quad (i = x,\ y,\ z)$$
$$\hat{P}_r \hat{J}_\mu \hat{P}_r^{-1} = \hat{J}_\mu, \quad (\mu = \pm 1,\ 0)$$

If a system consists of *N* subsystems, each of them being characterized by its own momentum, then the total momentum can be defined as:

$$\hat{\mathbf{J}} = \sum_{n=1}^{N} \hat{\mathbf{J}}(n)$$

The momenta related to different subsystems commute.

Matrix elements: $\langle J'M' | \hat{\mathbf{J}}^2 | JM \rangle = \delta_{JJ'} \delta_{MM'} J(J+1)$

Cartesian

$$\langle JM \pm 1 | \hat{J}_x | JM \rangle = \frac{1}{2} \sqrt{(J \pm M + 1)(J \mp M)},$$
$$\langle JM \pm 1 | \hat{J}_y | JM \rangle = \mp \frac{i}{2} \sqrt{(J \pm M + 1)(J \mp M)},$$
$$\langle JM | \hat{J}_z | JM \rangle = M.$$

Other matrix elements are zero.

Cyclic covariant

$$\langle JM \pm 1 | \hat{J}_{1\pm 1} | JM \rangle = \mp \sqrt{\frac{(J \pm M + 1)(J \mp M)}{2}},$$
$$\langle JM | \hat{J}_{10} | JM \rangle = M.$$

J must be non-negative integer or half-integer.

Clebsch-Gordan coefficients

Consider $(2j_1 + 1)(2j_2 + 1)$ functions $< \Omega_1 | j_1 m_1 > < \Omega_2 | j_2 m_2 > \equiv < \Omega_1\, \Omega_2 | j_1 m_1\, j_2 m_2 >$.

They form a representation of the rotation group, but not an irreducible one.

$$< \Omega_1\, \Omega_2 | j_1 m_1\, j_2 m_2 > = \sum_{j\,m} < \Omega_1\, \Omega_2 | j_1\, j_2 j\, m > < j_1\, j_2 j\, m | j_1 m_1\, j_2 m_2 > .$$

Clebsch-Gordan coefficient:

$$C^{j\,m}_{j_1 m_1\, j_2 m_2} = \langle j_1 m_1 j_2 m_2 \mid j_1 j_2 j m \rangle = \langle j_1 j_2 j m \mid j_1 m_1 j_2 m_2 \rangle$$

CG coefficients are chosen real and $C^{j\, j_1 - j_2}_{j_1 j_1\, j_2 - j_2} > 0$ (Condon-Shortley convention).

CG coefficients are non-zero, only if $m_1 + m_2 = m$, since $\hat{j}_{1z} + \hat{j}_{2z} = \hat{j}_z$.

The total momentum j takes values $|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 - 1, j_1 + j_2$, each value only once (the triangle rule).

Check: sum of dimensions of these IRs = $(2j_1 + 1)(2j_2 + 1)$.

Unitarity:

$$\sum_{m_1 m_2} C^{j\,m}_{j_1 m_1\, j_2 m_2} C^{j'\,m'}_{j_1 m_1\, j_2 m_2} = \delta_{j j'} \delta_{m m'},$$

$$\sum_{j(m)} C^{j\,m}_{j_1 m_1\, j_2 m_2} C^{j\,m}_{j_1 m_1'\, j_2 m_2'} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}.$$

3*jm*-symbols

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_3+m_3+2j_1} \frac{1}{\sqrt{2j_3+1}} C^{j_3 m_3}_{j_1 - m_1\, j_2 - m_2}$$

$$C^{j_3 m_3}_{j_1 m_1\, j_2 m_2} = (-1)^{j_1 - j_2 + m_3} \sqrt{2j_3+1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

If $m_1 + m_2 + m_3 \neq 0$ then the 3*jm*-symbol is zero.

Symmetry properties:

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \begin{pmatrix} b & c & a \\ \beta & \gamma & \alpha \end{pmatrix} = \begin{pmatrix} c & a & b \\ \gamma & \alpha & \beta \end{pmatrix} = (-1)^{a+b+c} \begin{pmatrix} a & c & b \\ \alpha & \gamma & \beta \end{pmatrix} = (-1)^{a+b+c} \begin{pmatrix} b & a & c \\ \beta & \alpha & \gamma \end{pmatrix} = (-1)^{a+b+c} \begin{pmatrix} c & b & a \\ \gamma & \beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{a+b+c} \begin{pmatrix} a & b & c \\ -\alpha & -\beta & -\gamma \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{a+b+c} \begin{pmatrix} \frac{b+c+\alpha}{2} & \frac{a+c+\beta}{2} & \frac{a+b+\gamma}{2} \\ a - \frac{b+c-\alpha}{2} & b - \frac{a+c-\beta}{2} & c - \frac{a+b-\gamma}{2} \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \begin{pmatrix} a & \frac{b+c-\alpha}{2} & \frac{b+c+\alpha}{2} \\ -b+c & \frac{b-c-\alpha}{2} - \gamma & \frac{b-c+\alpha}{2} + \gamma \end{pmatrix}$$

[Regge]

Calculation of Clebsch-Gordan coefficients (Racah‘s algorithm)

$|j_1\, m_1 = j_1\;\; j_2\, m_2 = j_2\rangle = |j_1\;\; j_2\; j = j_1 + j_2\;\; m = j_1 + j_2\rangle$, i.e., $C^{j_1+j_2\;\; j_1+j_2}_{j_1 j_1 j_2 j_2} = 1$.

Then apply to the both sides the lowering operator $\hat{J}^{(1)}_{-1} + \hat{J}^{(2)}_{-1} = \hat{J}_{-1}$

$$\hat{J}_{-1}\left| j_1 j_2\;\; j = j_1 + j_2\;\; m = j_1 + j_2 \right\rangle = -\sqrt{j_1 + j_2}\left| j_1 j_2\;\; j = j_1 + j_2\;\; m = j_1 + j_2 - 1 \right\rangle =$$

$$= -\sqrt{j_1 + j_2}\sum_{m_2 = j_2 - 1}^{j_2} C^{j_1+j_2\;\; j_1+j_2-1}_{j_1 j_1+j_2-1-m_2\;\; j_2 m_2}\left| j_1 m_1 = j_1 + j_2 - 1 - m_2\;\; j_2 m_2 \right\rangle =$$

$$= -\sqrt{j_1}\left| j_1 m_1 = j_1 - 1\;\; j_2 m_2 \right\rangle - \sqrt{j_2}\left| j_1 m_1 = j_1\;\; j_2 m_2 - 1 \right\rangle$$

and equate the coefficients in front of $|j_1\, m_1 = j_1 - 1\; j_2\, m_2 = j_2\rangle$ in the r.h.s. & in the l.h.s. ; then the same for $|j_1\, m_1 = j_1\; j_2\, m_2 = j_2 - 1\rangle$. Then we obtain

$$C^{j_1+j_2\;\; j_1+j_2-1}_{j_1 j_1-1\;\; j_2 j_2} = \sqrt{j_1/(j_1 + j_2)} \qquad C^{j_1+j_2\;\; j_1+j_2-1}_{j_1 j_1\;\; j_2 j_2-1} = \sqrt{j_2/(j_1 + j_2)}$$

and so on, until we calculate $C^{j_1+j_2\;\; m_1+m_2}_{j_1 m_1\;\; j_2 m_2}$

for all $m_1 + m_2$ up to $-(j_1 + j_2)$, where we must end up with a CG that is equal to

$(-1)^{j_1+j_2-j} = 1$ for $j = j_1 + j_2$.

Then we construct a linear combination of

$$| j_1\, m_1 = j_1 - 1\; j_2\, m_2 = j_2 > \text{ and } | j_1\, m_1 = j_1\; j_2\, m_2 = j_2 - 1 >,$$

which is orthogonal to $| j_1 \;\; j_2 \; j = j_1 + j_2 \;\; m = j_1 + j_2 - 1>$, normalize it to 1 and (later) choose the common sign according to the Condon-Shortley convention. Then we obtain explicitly $| j_1 \;\; j_2 \;\; j = j_1 + j_2 - 1 \;\; m = j_1 + j_2 - 1>$ and CG coefficients

$$C^{j_1+j_2-1 \;\; j_1+j_2-1}_{j_1 j_1 - 1 \;\; j_2 j_2} \qquad \text{and} \qquad C^{j_1+j_2-1 \;\; j_1+j_2-1}_{j_1 j_1 \;\; j_2 j_2 - 1}$$

We apply again the lowering operator and find all CG coefficients for $j = j_1 + j_2 - 1$. The same procedure applies for all j's down to $| j_1 - j_2 |$. The general form of the recurrence relation is

$$\Gamma_-(j, m) C^{j\; m-1}_{j_1 m_1 \; j_2 m_2} = \Gamma_-(j_1, m_1 + 1) C^{j\; m}_{j_1 m_1 + 1 \; j_2 m_2} + \Gamma_-(j_2, m_2 + 1) C^{j\; m}_{j_1 m_1 \; j_2 m_2 + 1}$$

$$\Gamma_-(j, m) = \sqrt{(j + m)(j - m + 1)}$$

Wigner *D*-functions (continued)

$$D^J_{MM'}(\alpha, \beta, \gamma) = e^{-iM\alpha} d^J_{MM'}(\beta)\, e^{-iM'\gamma}$$

$$d^J_{MM'}(\beta) = \langle JM | \exp\left(-i\beta \hat{J}_y\right) | JM' \rangle$$

$$d^J_{MM'}(\beta) = (-1)^{J-M'} [(J+M)!\,(J-M)!\,(J+M')!\,(J-M')!]^{1/2} \times$$

$$\times \sum_k (-1)^k \frac{\left(\cos\frac{\beta}{2}\right)^{M+M'+2k} \left(\sin\frac{\beta}{2}\right)^{2J-M-M'-2k}}{k!\,(J-M-k)!\,(J-M'-k)!\,(M+M'+k)!},$$

Sum over all non-negative *k* providing non-negative arguments of all factorials.

$d^J_{MM'}(\beta)$ can be expressed through special functions (the hypergeometric function or the Jacobi polynomials).

Addition of rotations.

The first rotation is defined by the Euler angles $\alpha_1, \beta_1, \gamma_1$ with respect to the **old** axes *x,y,z* (scheme B).
The second rotation is defined by the Euler angles $\alpha_2, \beta_2, \gamma_2$ with respect to the **old** axes *x,y,z* (scheme B).
The resulting rotation is defined by the Euler angles α, β, γ with respect to the **old** axes *x,y,z* (scheme B).

$$\sum_{M''=-J}^{J} D^J_{MM''}(\alpha_2, \beta_2, \gamma_2)\, D^J_{M''M'}(\alpha_1, \beta_1, \gamma_1) = D^J_{MM'}(\alpha, \beta, \gamma)$$

Orthogonality (follows from the properties of the special functions involved):

$$\int_0^{4\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma D^{J_2 *}_{M_2 M'_2}(\alpha, \beta, \gamma) D^{J_1}_{M_1 M'_1}(\alpha, \beta, \gamma) =$$

$$= \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{4\pi} d\gamma D^{J_2 *}_{M_2 M'_2}(\alpha, \beta, \gamma) D^{J_1}_{M_1 M'_1}(\alpha, \beta, \gamma) = \frac{16\pi^2}{2J_1 + 1} \delta_{J_1 J_2} \delta_{M_1 M_2} \delta_{M'_1 M'_2}$$

In physical applications J_1 and J_2 are usually either both integer or both half-integer. Then the orthogonality relation can be written as

$$\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma D^{J_2 *}_{M_2 M_2'}(\alpha, \beta, \gamma) D^{J_1}_{M_1 M'_1}(\alpha, \beta, \gamma) = \frac{8\pi^2}{2J_1 + 1} \delta_{J_1 J_2} \delta_{M_1 M_2} \delta_{M_1' M_2'}$$

$$\int_0^{\pi} d\beta \sin\beta \, d^J_{M M'}(\beta) \, d^{J'}_{M M'}(\beta) = \frac{2}{2J + 1} \delta_{J J'}$$

Completeness

$$\sum_{J=0,\,1/2,\,1,\,\dots}^{\infty} \sum_{M=-J}^{J} \sum_{M'=-J}^{J} \frac{2J+1}{16\pi^2} D^{J*}_{MM'}(\alpha_1, \beta_1, \gamma_1)\, D^{J}_{MM'}(\alpha_2, \beta_2, \gamma_2) = \delta(\alpha_1 - \alpha_2)\, \delta(\cos\beta_1 - \cos\beta_2)\, \delta(\gamma_1 - \gamma_2)$$

$$V_1: \quad 0 \leqslant \alpha < 4\pi, \quad 0 \leqslant \beta \leqslant \pi, \quad 0 \leqslant \gamma < 2\pi;$$
$$V_2: \quad 0 \leqslant \alpha < 2\pi, \quad 0 \leqslant \beta \leqslant \pi, \quad 0 \leqslant \gamma < 4\pi.$$

If $\displaystyle\iiint_{V_1(V_2)} d\alpha\, d\beta \sin\beta\, d\gamma\, |f(\alpha, \beta, \gamma)|^2 < \infty$ then

$$f(\alpha, \beta, \gamma) = \sum_{J=0,\,1/2,\,1,\,\dots}^{\infty} \sum_{M=-J}^{J} \sum_{M'=-J}^{J} a^{J}_{MM'} D^{J}_{MM'}(\alpha, \beta, \gamma)$$

$$a^{J}_{MM'} = \frac{2J+1}{16\pi^2} \iiint_{V_1(V_2)} d\alpha\, d\beta \sin\beta\, d\gamma\, f(\alpha, \beta, \gamma)\, D^{J*}_{MM'}(\alpha, \beta, \gamma)$$

Recurrence approach to the calculation of *D*-functions.

For a scalar ($J = 0$) the corresponding *D*-function ≡ 1.
The first non-trivial case is $J = ½$.

If $J = ½$, then $\hat{J}_y = ½\,\hat{\sigma}_y$, where $\hat{\sigma}_y$ is the Pauli matrix. Recalling that $\hat{\sigma}_y^{\,2} = 1$, we obtain

$d^{1/2}_{MM'}(\beta)$

M \ M'	1/2	− 1/2
1/2	$\cos\frac{\beta}{2}$	$-\sin\frac{\beta}{2}$
− 1/2	$\sin\frac{\beta}{2}$	$\cos\frac{\beta}{2}$

Check it!

We have an obvious expression

$$\sum_{\substack{M_1 M_2 \\ N_1 N_2}} C^{JM}_{J_1M_1J_2M_2} D^{J_1}_{M_1N_1}(\alpha,\ \beta,\ \gamma)\, D^{J_2}_{M_2N_2}(\alpha,\ \beta,\ \gamma)\, C^{J'N}_{J_1N_1J_2N_2} = \delta_{JJ'}\{J_1J_2J\}\, D^{J}_{MN}(\alpha,\ \beta,\ \gamma)$$

(II.*)

The symbol $\{J_1J_2J_3\} = \begin{cases} 1, \text{ if } & J_1 + J_2 + J_3 \text{ is integer and } |J_1 - J_2| \leqslant J_3 \leqslant J_1 + J_2 \\ 0 & \text{otherwise} \end{cases}$

$\{J_1,J_2,J_3\}=1$ if the additon of momenta makes sense and = 0 otherwise (triangle rule).

Check it!

$d^1_{MM'}(\beta)$

$M \backslash M'$	1	0	−1
1	$\frac{1+\cos\beta}{2}$	$-\frac{\sin\beta}{\sqrt{2}}$	$\frac{1-\cos\beta}{2}$
0	$\frac{\sin\beta}{\sqrt{2}}$	$\cos\beta$	$-\frac{\sin\beta}{\sqrt{2}}$
−1	$\frac{1-\cos\beta}{2}$	$\frac{\sin\beta}{\sqrt{2}}$	$\frac{1+\cos\beta}{2}$

Using Eq. (II.*), we can construct subsequently *D*-functions for all *J*, starting from *J* = ½ .

Using Eq. (II.*), try to prove that

$$\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma\, D^{J_3 *}_{M_3 M'_3}(\alpha, \beta, \gamma)\, D^{J_2}_{M_2 M'_2}(\alpha, \beta, \gamma)\, D^{J_1}_{M_1 M'_1}(\alpha, \beta, \gamma) =$$

$$= \frac{8\pi^2}{2J_3 + 1} C^{J_3 M_3}_{J_1 M_1 J_2 M_2} C^{J_3 M'_3}_{J_1 M'_1 J_2 M'_2}$$

$J_1 + J_2 + J_3$ is integer

Try to prove that for positive integer *J*

$$C^{J0}_{J010} = 0$$