XIII. Schwinger model for the angular momentum operator

Consider two independent bosonic modes described by annihilation/creation operator obeying the standard commutation relation:

$$[\hat{a},\hat{a}^{+}]=1, \quad [\hat{b},\hat{b}^{+}]=1, \quad [\hat{a},\hat{b}]=0, \quad [\hat{a},\hat{b}^{+}]=0$$

Then one can show that the operators

$$\hat{J}_x=\frac{\hat{a}^{+}\hat{b}+\hat{b}^{+}\hat{a}}{2}, \quad \hat{J}_y=\frac{\hat{a}^{+}\hat{b}-\hat{b}^{+}\hat{a}}{2i}, \quad \hat{J}_z=\frac{\hat{a}^{+}\hat{a}-\hat{b}^{+}\hat{b}}{2}$$

satisfy the commutation relation for the components of the angular momentum operator

$$[\hat{J}_x,\hat{J}_y]=i\hat{J}_z, \quad [\hat{J}_y,\hat{J}_z]=i\hat{J}_x, \quad [\hat{J}_z,\hat{J}_x]=i\hat{J}_y.$$

Also $[\hat{J}_\ell,\hat{J}^2]=0, \ \ell=x,y,z,$ where

$$[\hat{N},\hat{J}_\ell]=0, \quad \ell=x,y,z$$

therefore we can replace the atom-number operator with its eigenvalue *N*

$$\hat{J}^2\equiv\hat{J}_x^2+\hat{J}_y^2+\hat{J}_z^2=\left(\frac{\hat{N}}{2}+1\right)\frac{\hat{N}}{2}, \qquad \hat{N}\equiv\hat{a}^{+}\hat{a}+\hat{b}^{+}\hat{b}$$

Eigenvalues *N* of $\hat{N}$ are non-negative integers $\Rightarrow$ the momentum $J = N/2$ is half-integer ≥ 0.

Cyclic components $\hat{J}_{\pm 1} = \mp \frac{1}{\sqrt{2}} \left(\hat{J}_x \pm i \hat{J}_y \right), \quad \hat{J}_0 = \hat{J}_z$

$\hat{J}_{+1} = -\frac{\hat{a}^{+}\hat{b}}{\sqrt{2}}$ raises M by 1,

$\hat{J}_{-1} = \frac{\hat{b}^{+}\hat{a}}{\sqrt{2}}$ lowers M by 1, where M is an eigenvalue of $\hat{J}_0$.

Holstein–Primakoff transformation

Mapping of quantum ang.momentum to bosonic annihilation/creation operators. Consider $| J, M = +J >$ as a vacuum state and, respectively, $m = J - M$ as the number of excitations. Introduce formally bosonic operators $\hat{c},\ \hat{c}^{\dagger},\ [\hat{c}, \hat{c}^{\dagger}] = 1$

$$|J, M = J - m\rangle = (m!)^{-1/2} c^{\dagger\, m} |\text{vac}\rangle$$

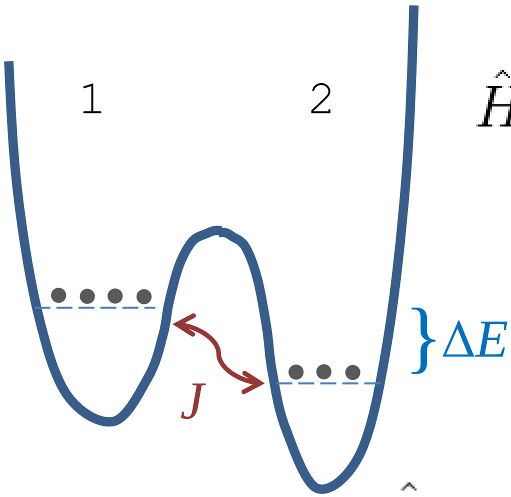
Then, recalling the expression for the matrix elements of cyclic components of $\hat{\mathbf{J}}$ we obtain

$$\hat{J}_0 = J - \hat{c}^{\dagger}\hat{c}\,, \quad \hat{J}_{+1} = -\sqrt{J}\sqrt{1 - \frac{\hat{c}^{\dagger}\hat{c}}{2J}}\,\hat{c}, \quad \hat{J}_{-1} = +\sqrt{J}\,\hat{c}^{\dagger}\sqrt{1 - \frac{\hat{c}^{\dagger}\hat{c}}{2J}}$$

This transformation is especially convenient for the small number of excitations, $m << J$, where one can expand these expressions in Taylor series in $\frac{\hat{c}^{\dagger}\hat{c}}{2J}$.

XIV. Quantum models to be mapped on angular-momentum problems

(XIV.1) Two-mode Bose-Hubbard model (ultracold atoms in a double well potential – the simplest, 2-mode description)



$$\hat{H} = U_1 \hat{N}_1^2 + U_2 \hat{N}_2^2 + \frac{\Delta E}{2}(\hat{N}_1 - \hat{N}_2) - J(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)$$

$$\hat{N}_j = \hat{a}_j^\dagger \hat{a}_j, \qquad j = 1, 2$$

$$\hat{N} = \hat{N}_1 + \hat{N}_2, \qquad [\hat{N}, \hat{H}] = 0$$

$$\hat{S}_z = \frac{\hat{N}_1 - \hat{N}_2}{2}, \qquad \hat{S}_x = \frac{\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1}{2}, \qquad \hat{S}^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right)$$

$$\hat{H} = (U_1 + U_2)\frac{N^2}{4} + (U_1 + U_2)\hat{S}_z^2 + [\Delta E + N(U_1 - U_2)]\hat{S}_z - 2J\hat{S}_x$$

(XIV.2) Dicke model

A two-level system consisting of two states, ground |g> and excited |e>, is formally equivalent to a (pseudo)spin s = 1/2.

The raising operator

$$\hat{\sigma}^{+} = \hat{s}_x + i\,\hat{s}_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

transformes |g> into |e>,

the lowering operator

$$\hat{\sigma}^{-} = \hat{s}_x - i\,\hat{s}_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

transforms |e> into |g>.

The operator of the population difference

$$\hat{\sigma}_z = 2\,\hat{s}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hamiltonian of N two-level system with a single electromagnetic mode (practically, with a cavity mode). We denote the photon annihilation operator by $\hat{a}$.
If we assume that the atom-photon coupling constant is the same for all atoms, this Hamiltonian reads as (we set Planck´s constant $\hbar = 1$)

$$\hat{H} = \omega \hat{a}^{\dagger}\hat{a} + \sum_{i=1}^{N} [\omega_0 \hat{s}_{z\,i} + 2g(\hat{a}^{\dagger} + \hat{a})\hat{s}_{x\,i}]$$

Sum of individual spin operators yield the collective spin operator:

$$\hat{H} = \omega \hat{a}^{\dagger}\hat{a} + \omega_0 \hat{S}_z + 2g(\hat{a}^{\dagger} + \hat{a})\hat{S}_x$$

Since the e.m.-mode is close to the resonance, $\omega \approx \omega_0$, we can use the rotating wave approximation (RWA):

$$\hat{H} = \omega \hat{a}^{\dagger}\hat{a} + \omega_0 \hat{S}_z + g(\hat{a}^{\dagger}\hat{S}^{-} + \hat{a}\,\hat{S}^{+})$$

What is the integral of motion of this Hamiltonian?

Note: the same coupling constant for all atoms may be attained for a ring (running-wave) cavity; the phase factors exp (i**k**$\mathbf{r}_j$) for different atoms can be included into the definition of $|e\rangle_j$.
The use of the Holstein-Primakoff transformation reduces the Hamiltonian to one for two bosonic fields (atomic excitations and phonons). How this bosonic Hamiltonian looks if the number of at.excitations + the number of photons << N? In the case of small number of excitations and phonons write the Hamiltonian in the case of non-equal coupling constant (each atom possessing its own g_j).

But N pseudospins $s = 1/2$ may be summed in different ways.
If they form a fully simmetrized state, i.e., characterized by the Young diagram $\{N\}$, then we obtain max.possible collective spin $S = N/2$.
In a general case, for the Young diagram $\{N, N-m\}$, where $m \le N/2$, we obtain $S = N/2 - m$.
In particular, for an even N and $m = N/2$ (the Young diagram consisting of two rows of the equal length) $S = 0$.

The rate Γ of photon emission into the cavity mode is proportional to $\langle \hat{S}^+ \hat{S}^- \rangle$.

If (almost) all atoms are in the |e> state, $\langle \hat{S}_z \rangle \approx S$, then $\Gamma \propto S$.
When in the course of evolution, almost half of the atoms decayed into the state |g>, i.e.,

when $\langle \hat{S}_z \rangle \approx 0$, we obtain $\Gamma \propto S^2$.
The states with $\{\lambda\} = \{N\}$ and, hence, $S = N/2$ are called Dicke states. They are charcterized by the maximum possible photon emission rate

$\langle \hat{S}_z \rangle \approx N/2 \Rightarrow \Gamma \propto N$ Atoms emit photons independently.

$\langle \hat{S}_z \rangle \approx 0 \Rightarrow \Gamma \propto N^2$ Collective (enhanced) emission – superradiance.

The opposite limit: states with $\{\lambda\} = \{N/2, N/2\}$ for even N and, hence, $S = 0$, do not emit into the cavity mode at all. Do they emit into other modes (side modes)? Why?