XII. Addition of many identical spins

(XII.1). Symmetric group

Symmetric group S_n is the group of all possible permutations of n objects.
In total $n!$ elements (including identity operation).
Each permutation is a product of a certain finite number of *pairwise* transpositions. For a given permutation, this number is always even or always odd (the same permutation can be achieved by different sequencies of transpositions). Therefore, we can speak about even and odd permutations.
The sign of a permutation P:

$$\operatorname{sgn}(P) = \begin{cases} +1 \quad , \quad P \text{ even} \\ -1 \quad , \quad P \text{ odd} \end{cases}$$

Irreducible representations of the symmetric group

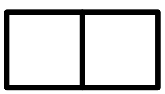
IRs of S_n correspond to conjugacy classes of S_n and are labelled by a partition of the integer, positive number n.

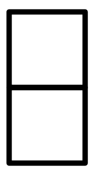
$$\{\lambda\} \equiv \{\lambda_1, \lambda_2, \ldots, \lambda_r\}, \quad n = \lambda_1 + \lambda_2 + \ldots + \lambda_r.$$

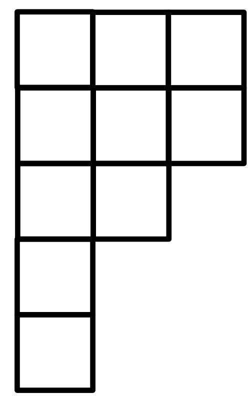
We order these integer number as

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0.$$

This partition can be represented graphically by so-called Young diagrams.
A Young diagram contains λ_j boxes in its jth row.
Few examples:

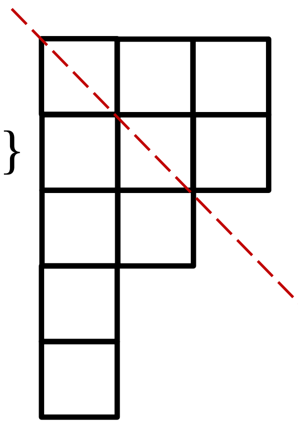
{2} 

{1,1} 

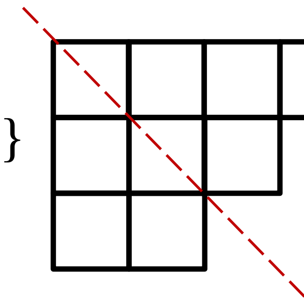
{3,3,2,1,1} 

Conjugate diagrams: rows ↔ colums, or, equivalently, reflection with respect to the diagonal .

$\{\lambda\}=\{3,3,2,1,1\}$



$\{\tilde{\lambda}\}=\{5,3,2\}$



Conjugation is denoted by ~

Young tableau: a Young diagram (*n* boxes) filled with integer numbers 1,2, ... , *n*.
Standard Young tableau: numbers in each row and in each column are placed in the increasing order.
Example: all standard Young tableaux for $\{\lambda\}=\{3,2\}$.

1	2	3
4	5	

1	2	4
3	5	

1	2	5
3	4	

1	3	4
2	5	

1	3	5
2	4	

The dimension $d_{\{\lambda\}}$ of an IR of S_n characterized by the Young diagram $\{\lambda\}$ is equal to the number of corresponding standard Young tableaux.
The dimensions of IRs characterized by conjugate Young diagrams are the same,

$$d_{\{\lambda\}} = d_{\{\tilde{\lambda}\}}$$

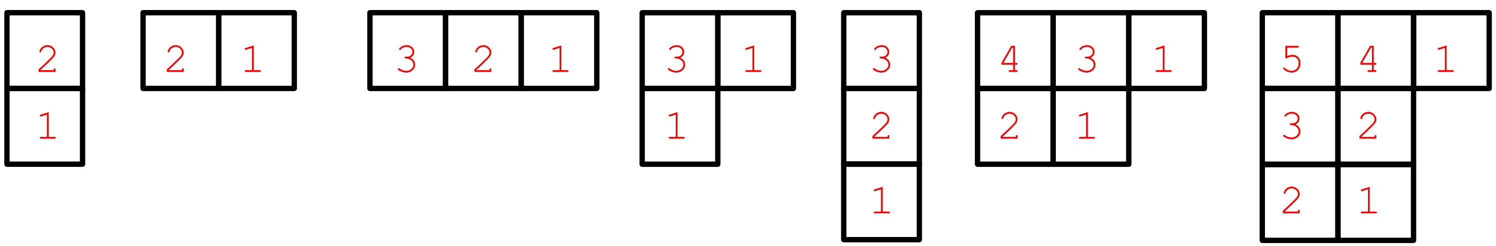
Note that

$$\sum_{\{\lambda\}} d_{\{\lambda\}}^{2} = n!$$

Hook length $h(j)$ of the jth box of a Young diagram:

$h(j) =$ number of boxes to the right in the row +
+ number of boxes tbelow in the column + 1

Hook length for some $\{\lambda\}$



An easier way to calculate $d_{\{\lambda\}}$ is given by a theorem:

$$d_{\{\lambda\}} = \frac{n!}{\prod_{j=1}^{n} h(j)}$$

(XII.2). Systems of identical particles.

Consider n identical particles with the spin s.
Simultaneous permutation of both co-ordinates $\mathbf{r}_j$ and spin variables σ_j of each pair of particles multiplies the n-particle wave function by +1 if s is integer (bosons, B) and by – 1 if s is half-integer (fermions, F).
A wave function satisfying this symmetry requirement can be constructed in many ways, corresponding to various Young diagrams

$$\Psi_B(\mathbf{r}_1,\ldots,\mathbf{r}_n;\sigma_1,\ldots,\sigma_n) = \sum_\tau \Phi_{\{\lambda\}\tau}(\mathbf{r}_1,\ldots,\mathbf{r}_n)\mathrm{X}_{\{\lambda\}\tau}(\sigma_1,\ldots,\sigma_n)$$

$$\Psi_F(\mathbf{r}_1,\ldots,\mathbf{r}_n;\sigma_1,\ldots,\sigma_n) = \sum_\tau \Phi_{\{\tilde{\lambda}\}\tilde{\tau}}(\mathbf{r}_1,\ldots,\mathbf{r}_n)\mathrm{X}_{\{\lambda\}\tau}(\sigma_1,\ldots,\sigma_n)$$

Here sum is taken over all $d_{\{\lambda\}}$ standard Young tableaux, where we use spin variables σ_j instead of numbers j = 1,2, ... , n to fill the boxes, when we construct the spin part X. The co-ordinate part is obtained by replacing the spin variables with the co-ordinates and, in the case of fermions only, by conjugating the Young tableau.
We focus on the construction of the spin part X of the wave function, the co-ordinate part being built in a similar way.

1. Choose a certain standard Young tableau, like that:

σ_1	σ_3	σ_6
σ_2	σ_4	
σ_5	σ_7	

2. Construct a product of single spin wave functions $\chi(\sigma_1,\ldots,\sigma_n)=\prod_{j=1}^{n}\langle\sigma_j\,|\,s\,m_s^j\rangle$

where $m_s^j = -s, -s+1, \ldots, s-1, s$ is the projection of the spin of the jth particle to the quantization axis.

3. Apply to this product a Young symmetrizer.

For a given Young tableau we select from all $n!$ permutations only those operations (including the identity operation!), which do not permute spin variables belonging to different rows. We denote these permutations by P_r.

How many are there P_r´s?

Also we select permutations (including identity!), which do not permute spin variables belonging to different columns, denoting them by P_c.

How many are there P_c´s?

Then, summing by all possible P_r and P_c, we define the Young symmetrizer:

$$Y_{\{\lambda\}\tau} = \sum_{P_r, P_c} \mathrm{sgn}(P_c) P_r P_c$$

Spin wave function

$$X_{\{\lambda\}\tau}(\sigma_1,\ldots,\sigma_n) = Y_{\{\lambda\}\tau}\chi(\sigma_1,\ldots,\sigma_n)$$

The maximum number of row in $\{\lambda\}$ is $2s + 1$, since antisymmetrization over variables in columns containing more than $2s + 1$ will require antisymmetrization over arguments σ_j and σ_i of two functions $< \sigma_j \mid s\ m_s >$ and $< \sigma_i \mid s\ m_s >$ with the same m_s . The result of the antisymmetrization will then be identically zero.
In particular, for spin-1/2 particles, such as electrons, the spin wave function can be characterized by Young diagrams with 1 row (fully symmetric) or 2 rows only.

It is possible to calculate, which values of the total spin S built from n identical spins s correspond to a given Young diagram.
If the total spin S appears $W_{S\{\lambda\}}$ times for the $\{\lambda\}$-type wave function than, obviously

$$\sum_{\{\lambda\}}\sum_{S} W_{\{\lambda\}S}(2S+1) = (2s+1)^n$$

For $s = ½$, unlike other (non-zetro) values of particle spin, there is a one-to-one correspondence between $\{\lambda\}$ and S.

The rule:

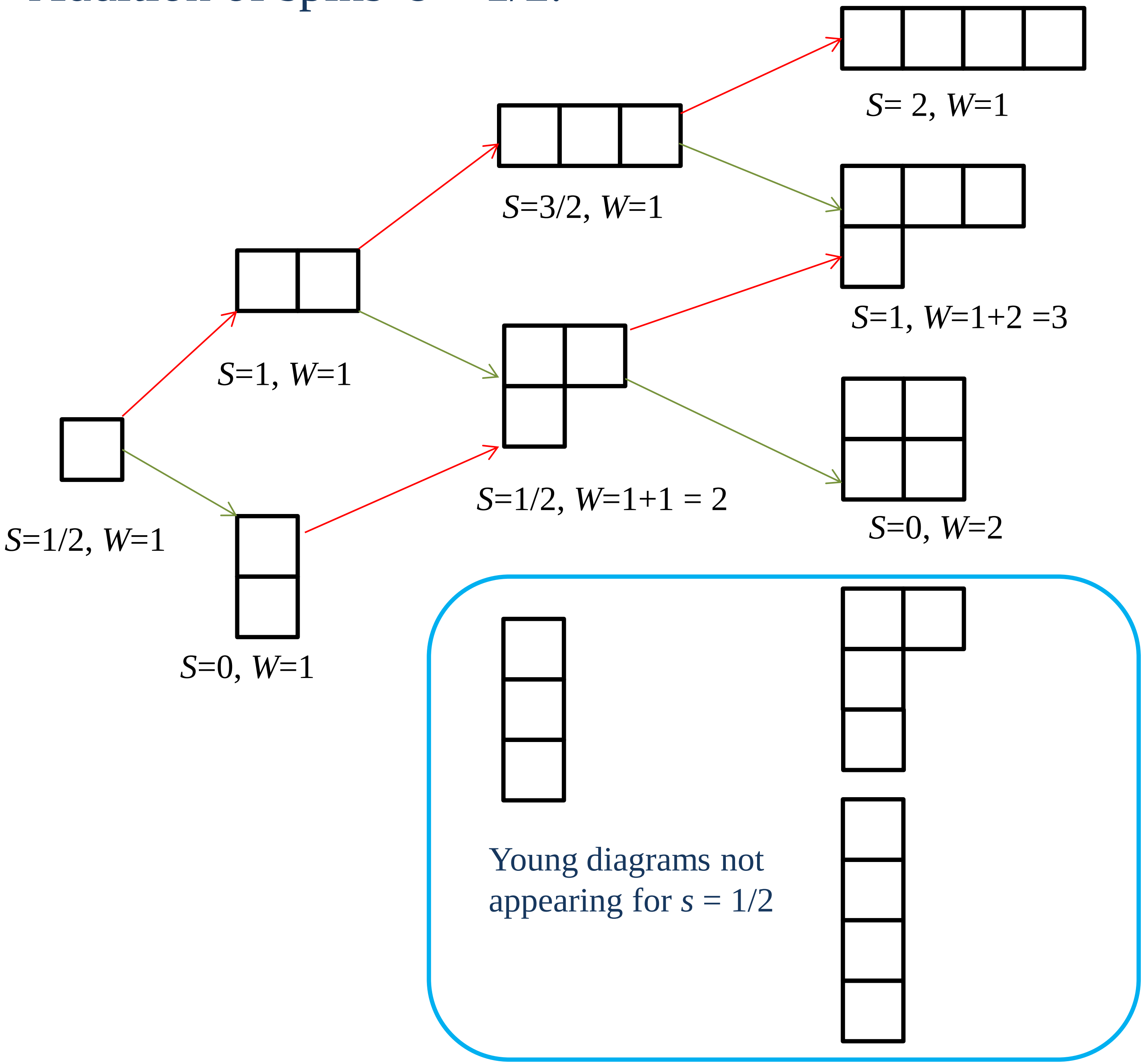
If we add to a state of n spin-½ particles with the total spin S and $\{\lambda\} = \{\lambda_1, \lambda_2\}$, $\lambda_1 + \lambda_2 = n$, a new particle of the same kind, then we obtain the states:

1) $\{\lambda_1+1, \lambda_2\}, S + ½$,

2) $\{\lambda_1, \lambda_2+1\}, S – ½$. The variant (2) is possible for $\lambda_1 > \lambda_2$ only.

Recall that W shows how many times the total spin S appears for a given $\{\lambda\}$.

Addition of spins s = 1/2.



...and so on.

For s > 1/2, for a given {λ}
W may be non-zero for different S

Explicit construction of wave functions of many spins $s = ½$

For each particle two states: „spin up“ and „spin-down“.

Spin-lowering operator for jth particle:

$$\sigma_j^- |\uparrow\rangle_j = |\downarrow\rangle_j, \quad \sigma_j^- |\downarrow\rangle_j = 0$$

$$\sigma_j^- |\uparrow\rangle_i = 0, \quad \sigma_j^- |\downarrow\rangle_i = 0 \quad \text{for } i \neq j$$

Collective spin-lowering operator:

$$\sigma^- = \sum_{j=1}^{n} \sigma_j^-$$

The highest possible $S = n/2$ corresponds to the fully symmetrized state (Young diagram {n} consists of a single row).

The highest possible collective spin projection $M = +S = +n/2$: all atoms in the spin-up state.

$$|\{n\}, S = n/2, M = +n/2\rangle = \prod_{j=1}^{n} |\uparrow\rangle_j$$

Applying the collective spin-lowering operator, we decrease M by 1:

$$|\{n\}, S = n/2, M = (n/2) - 1\rangle = \frac{1}{\sqrt{n!}} \sigma^- |\{n\}, S = n/2, M = n/2\rangle =$$

$$= \frac{1}{\sqrt{n!}} \sum_{i=1}^{n} |\downarrow\rangle_i \prod_{j \neq i} |\uparrow\rangle_j$$

But there are n linearly independent states with $M = (n/2) - 1$: one spin down, all other spins up:

$$\left|\downarrow\right\rangle_i \prod_{j\neq i}\left|\downarrow\right\rangle_j, \quad i = 1, 2, \ldots, n.$$

From them we can construct $(n - 1)$ linearly independent functions orthogonal to

$$\left|\{n\}, S = n/2, M = +n/2\right\rangle = \prod_{j=1}^{n}\left|\uparrow\right\rangle_j$$

These $(n - 1)$ functions are to be identified as different realizations of $S = (n/2) - 1$ for the Young diagram $\{n - 1, 1\}$.

We apply σ^- once again and obtain n functions for the states with $S = n/2$ and $S=(n/2) - 1$ with the total spin projection $M = (n/2) - 2$.

But there are $n(n - 1)/2$ lin.independent states with 2 spins down and $(n - 1)$ spins up. By orthogonalizing them to the already defined states, we obtain

$n(n - 1)/2 - n = n(n - 3)/2$ states for $S = (n/2) - 2$ and $\{\lambda\} = \{n - 2, 2\}$.

$\{\lambda\} = \{n - m, m\}$, where m is integer and $m \leq n/2$ corresponds to $S = (n/2) - m$.

Calculate $W_{\{\lambda\}, S}$ for $\{\lambda\} = \{n - m, m\}$, $S = (n/2) - m$.