VIII. Atom in an oscillating magnetic field

Recall that $\vec{e}_{\pm} = \mp \frac{1}{\sqrt{2}}(\vec{e}_x \pm i\vec{e}_y)$ $\quad \vec{e}_0 = \vec{e}_z$

$$A_\mu = \vec{A}\cdot\vec{e}_\mu$$

$$A_{\pm 1} = \mp \frac{1}{\sqrt{2}}(A_x \pm i A_y) \qquad A_0 = A_z$$

$$\vec{A}\cdot\vec{B} = \sum_{\mu=-1}^{+1} A_\mu B^\mu = \sum_{\mu=-1}^{+1} (-1)^\mu A_\mu B_{-\mu}$$

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d.c. magnetic field $\vec{B}_{d.c.}$ defines the direction of the axis z $\Rightarrow$ Zeeman shift (small field)

a.c. magnetic field

$$\vec{B}(t) = B_{0x'}\vec{e}_{x'}\cos\omega t + B_{0y'}\vec{e}_{y'}\cos(\omega t + \varphi)$$

Arbitrary (in general, ellyptic) polarization.

Propagation direction of the e.-m. wave defines the axis z' of the rotated system of co-ordinates

Intensity of an e.-m. wave propagating in vacuum

$$I = c\left(\frac{\overline{\varepsilon_0 \vec{E}^{\,2}}}{2} + \frac{\overline{\vec{B}^{\,2}}}{2\mu_0}\right),$$ where $\overline{\ldots}$ – time average (over a period $2\pi/\omega$)

Both terms in brackets are equal

$$I = \frac{c}{\mu_0}\overline{\vec{B}^{\,2}} = \frac{c}{2\mu_0}\left(B_{0x'}^2 + B_{0y'}^2\right)$$

Since $B_{0z'} \equiv 0$, we can express the intensity in the rotationally invariant form

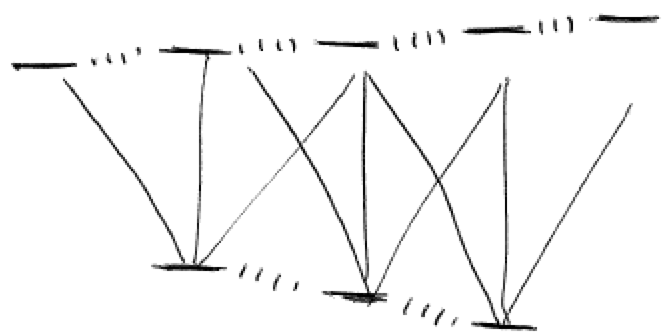
$$I = \frac{c}{2\mu_0}\sum_{i=x,y,z} B_{0i}^2 = \frac{2c}{\mu_0}\sum_{\mu=-1}^{+1}(-1)^{\mu}\underbrace{|B_{0\mu}|^2}_{\text{in cyclic coordinates}}$$

where the cyclic components are introduced via

$$\mathbf{B}\,(t) = (\,B_{-1}\,\mathbf{e}^{+1} + B_{+1}\,\mathbf{e}^{-1}\,)\exp\,(-\,\mathrm{i}\,\omega t\,) + \text{c.c.}$$

We have to find the transition matrix elements for ^{87}Rb

$F' = 2$



$F = 1$

— microwave transition (MW)

...... radiofrequency transition (RF)

$$H_{int} = -\hat{\vec{\mu}}\,\vec{B} = -\sum_{\lambda=-1}^{+1} (-1)^{\lambda}\,\hat{\mu}_{\lambda}\,B_{-\lambda}$$

Transformation of coordinates:

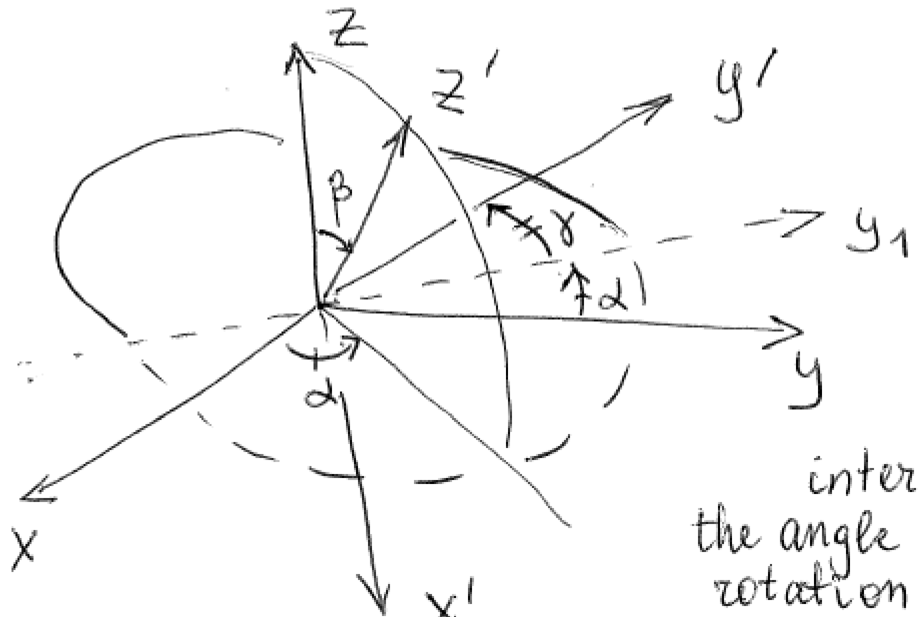
from $S \to S'$

$$B_{\lambda'} = \sum_{\lambda=-1}^{+1} B_\lambda \, D^1_{\lambda\lambda'}(\alpha,\beta,\gamma)$$

from $S' \to S$

$$B_\lambda = \sum_{\lambda'=-1}^{+1} \mathcal{D}^{1\,*}_{\lambda\lambda'}(\alpha,\beta,\gamma)\, B_{\lambda'}$$

Recall unitarity relations for $\mathcal{D}^J_{MM'}(\alpha,\beta,\gamma)$!



Finding Euler angles:

1) angle between z and $z' = \beta$
2) intersextions of the planes (z, z') and (x, y) defines angle α = Angle between x and this intersection line, or, equivalently, the angle between the intermediate rotation axis y_1 and (old) y
3) angle between y' and $y_1 = \gamma$

$$B_{\pm 1'}(t) = \mp \frac{1}{\sqrt{2}} \left[B_{0x'} \cos \omega t \pm i B_{0y'} \cos(\omega t + \varphi) \right]$$

$$B_{0'}(t) = 0$$

$$B_{+1}(t) = e^{i\alpha} \left[d^1_{11}(\beta) e^{i\gamma} B_{+1'}(t) + d^1_{1-1}(\beta) e^{-i\gamma} B_{-1'}(t) \right]$$

$$B_0(t) = \left[d^1_{01}(\beta) e^{i\gamma} B_{+1'}(t) + d^1_{0-1}(\beta) e^{-i\gamma} B_{-1'}(t) \right]$$

$$B_{-1}(t) = e^{-i\alpha} \left[d^1_{-11}(\beta) e^{i\gamma} B_{+1'}(t) + d^1_{-1-1}(\beta) e^{-i\gamma} B_{-1'}(t) \right]$$

In the new
coord. system with
z along <u>constant</u> $\vec{B}_{dc}$

For every $\lambda = -1, 0, +1$ we can write

$$B_\lambda(t) = B_\lambda^{(+)} e^{-i\omega t} + B_\lambda^{(-)} e^{+i\omega t}$$

B_λ is a "non-tilded" IR tensor of the rank 1.

$$(B_\lambda)^* = (-1)^{-\lambda} B_{-\lambda}$$

Hence, $B^{(+)}_\lambda = (-1)^{+\lambda} \left(B^{(-)}_{-\lambda}\right)^*$.

This can be checked directly, using the properties

$$d^J_{MM'}(\beta) = (-1)^{M-M'} d^J_{-M\,-M'}(\beta) = (-1)^{M-M'} d^J_{M'M}(\beta) = d^J_{-M'-M}(\beta)$$

$$\hat{H}_{int} = -\hat{\vec{\mu}}\,\vec{B} = -\sum_{\lambda=-1}^{+1} (-1)^\lambda \hat{\mu}_\lambda B_{-\lambda} = -\sum_{\lambda=-1}^{+1} (-1)^\lambda \hat{\mu}_{-\lambda} B_\lambda$$

$$\langle JIF'M_F' | \hat{H}_{int} | JIFM_F \rangle =$$

$$= -\sum_{\lambda=-1}^{+1} (-1)^\lambda \langle JIF'M_F' | \hat{\mu}_\lambda | JIFM_F \rangle B_{-\lambda}(t) =$$

$$= -\sum_{\lambda=-1}^{+1} (-1)^\lambda \langle JIF'M_F' | \hat{\mu}_{-\lambda} | JIFM_F \rangle B_\lambda(t)$$

To be definite, consider ^{87}Rb, where the F = 2 state is above the F = 1 state;

$\hbar\omega_{hfs}$ = hyperfine splitting energy.

$$i\frac{\partial}{\partial t}|\Psi\rangle = \frac{1}{\hbar}\left\{\sum_{F}\sum_{M_F}\left[\hbar\omega_{hfs}\delta_{F2} + E_{M_F}(F)\right]|FM_F\rangle\langle FM_F| + \right.$$

$$\left. + \hat{H}_{int}\right\}|\Psi\rangle$$

Zeeman energy in a weak constant magnetic field; F and M_F are good quantum numbers.

$$|\Psi\rangle = \sum_{F}\sum_{M_F}\psi_{FM_F}(t)|JIFM_F\rangle$$

$$i\frac{\partial}{\partial t}\psi_{FM_F} = \left(\omega_{hfs} + \hbar^{-1}E_{M_F}(F)\right)\psi_{FM_F} +$$

$$+\sum_{FM_F}\sum_{\lambda=-1}^{+1}\left(-\frac{1}{\hbar}\right)\langle \mathbf{JI}FM_F|\hat{\mu}_\lambda|\mathbf{JI}F'M_F'\rangle\cdot(-1)^\lambda\left[B_{-\lambda}^{(+)}e^{-i\omega t} + B_{-\lambda}^{(-)}e^{i\omega t}\right]\psi_{F'M_F'}$$

MICROWAVE TRANSITIONS

$\omega \approx \omega_{hfs}$, the hyperfine components are coupled

$$\psi_{F=1\,M_F} \equiv \phi_{F=1\,M_F}$$

$$\psi_{F=2\,M_F} \equiv \phi_{F=2\,M_F}\,e^{-i\omega t}$$

Then, discarding rapidly [$\sim \exp(\pm i\omega t)$ or $\sim \exp(\pm 2i\omega t)$] terms, we arrive at the system of equations with time-independent coefficients in the r.h.s.

$$i\frac{\partial}{\partial t}\phi_{F=2\,M_F} = \left(\omega_{hfs} - \omega + \hbar^{-1}E_{M_F}(F=2)\right)\phi_{F=2\,M_F} +$$

$$+\sum_{\lambda=-1}^{+1}\frac{(-1)^{\lambda+1}}{\hbar}\langle \mathbf{J}\mathbf{I}F=2\,M_F|\hat{\mu}_\lambda|\mathbf{J}IF'=1\,M_F-\lambda\rangle B_{-\lambda}^{(+)}\,\phi_{F'=1\,M_F-\lambda}$$

$$i\frac{\partial}{\partial t}\phi_{F=1\,M_F} = \frac{E_{M_F}(F=1)}{\hbar}\phi_{F=1\,M_F} +$$

$$+\sum_{\lambda=-1}^{+1}\frac{(-1)^{\lambda+1}}{\hbar}\langle JIF=1\,M_F|\hat{\mu}_\lambda|JIF'=2\,M_F-\lambda\rangle B_{-\lambda}^{(-)}\,\phi_{F'=2\,M_F-\lambda}$$

This system of equations is Hamiltonian, because

$$B_\lambda^{(+)} = (-1)^\lambda\left(B_{-\lambda}^{(-)}\right)^*$$

This set of equations represents the rotating wave approximation (RWA).

RADIOFREQUENCY TRANSITIONS

$$\omega \simeq |\Delta E_{Zeeman}/\hbar|, \text{ where } \Delta E_{Zeeman} = E_{M_{F+1}}(F) - E_{M_F}(F)$$

is the difference of Zeeman shifts of two adjacent magnetic sublevels.

The hyperfine sublevels with $F=2$ and $F=1$ are off-resonance, their coupling via a.c. magn. field $\sim e^{\pm i\omega t}$ can be neglected.

In general, RF intensity may be high enough to provide $\mu_B B \sim \hbar\omega$. Then the rotating wave approximation does not apply, and we have

$$i\frac{\partial}{\partial t}\psi_{FM_F} = \left(\omega_{hfs}\delta_{F2} + \hbar^{-1}E_{M_F}(F)\right)\psi_{FM_F} +$$

$$+\sum^{+1} + \frac{1}{\hbar}\langle JIFM_F|\hat{\mu}_{+1}|JIFM_F-1\rangle\left(B_{-1}^{(+)}e^{-i\omega t} + B_{-1}^{(-)}e^{i\omega t}\right)\psi_{FM_F-1}$$

$$+\frac{1}{\hbar}\langle JIFM_F|\hat{\mu}_{-1}|JIFM_F+1\rangle\left(B_{+1}^{(+)}e^{-i\omega t} + B_{+1}^{(-)}e^{i\omega t}\right)\psi_{FM_F+1}$$

Write explicitly the equations for RF transitions in the rotating wave approximation.