

# IV. The Wigner-Eckart theorem

For any irreducible tensor operator

$$\langle n' j' m' | \hat{\mathfrak{M}}_{k\kappa} | n j m \rangle = (-1)^{j'-m'} \begin{pmatrix} j' & k & j \\ -m' & \kappa & m \end{pmatrix} \langle n' j' || \hat{\mathfrak{M}}_k || n j \rangle = (-1)^{2k} C_{j m k \kappa}^{j' m'} \frac{\langle n' j' || \hat{\mathfrak{M}}_k || n j \rangle}{\sqrt{2j' + 1}}$$

The dependence on the projections  $m, m', \kappa$  is reduced to the Clebsch-Gordan coefficient (or, equivalently, to a  $3jm$ -symbol).

$\langle n' j' || \hat{\mathfrak{M}}_k || n j \rangle$  is called **reduced matrix element** and does not depend on the orientation of the co-ordinate system.

$n, n'$  are all quantum numbers other than  $j, m$  (e.g., radial).

## Proof (1st variant)

$$[\hat{J}_{\pm 1}, \mathfrak{M}_{k\kappa}] = \mp \sqrt{\frac{k(k+1) - \kappa(\kappa \pm 1)}{2}} \mathfrak{M}_{k\kappa \pm 1} \quad \text{by definition of an IR tensor}$$

$$\langle n' j' m' | [\hat{J}_{\pm 1}, \mathfrak{M}_{k\kappa}] | n j m \rangle = \mp \langle n' j' m' | \sqrt{\frac{k(k+1) - \kappa(\kappa \pm 1)}{2}} \mathfrak{M}_{k\kappa \pm 1} | n j m \rangle$$

$$\langle n' j' m' | \hat{J}_{\pm 1} = -(\hat{J}_{\mp 1} | n' j' m' \rangle)^\dagger$$

$$\begin{aligned} \sqrt{k(k+1) - \kappa(\kappa \pm 1)} \langle n' j' m' | \mathfrak{M}_{k\kappa \pm 1} | n j m \rangle &= \\ &= \sqrt{j'(j'+1) - m'(m' \mp 1)} \langle n' j' m' \mp 1 | \mathfrak{M}_{k\kappa} | n j m \rangle - \\ &\quad - \sqrt{j(j+1) - m(m \pm 1)} \langle n' j' m' | \mathfrak{M}_{k\kappa} | n j m \pm 1 \rangle \end{aligned}$$

$$\begin{aligned} \Gamma_{\pm}(k, \kappa) \langle n' j' m' | \mathfrak{M}_{k\kappa \pm 1} | n j m \rangle + \Gamma_{\pm}(j, m) \langle n' j' m' | \mathfrak{M}_{k\kappa} | n j m \pm 1 \rangle &= \\ = \Gamma_{\pm}(j', m' \mp 1) \langle n' j' m' \mp 1 | \mathfrak{M}_{k\kappa} | n j m \rangle \end{aligned}$$

$$\Gamma_{\pm}(j, m) = \sqrt{j(j+1) - m(m \pm 1)}$$

The same recurrence relations that define CG coefficients in Racah's method.

And, additionally, for  $m + \kappa - 1 = j'$

$$\Gamma_{-}(k, \kappa) C_{j m k \kappa - 1}^{j' j'} + \Gamma_{-}(j, m) C_{j m - 1 k \kappa}^{j' j'} = 0$$

that follows from

$$C_{\alpha \alpha \beta \beta}^{c c} = \delta_{\alpha + \beta, c} (-1)^{a - \alpha} \left[ \frac{(2c + 1)! (a + b - c)! (a + \alpha)! (b + \beta)!}{(a + b + c + 1)! (a - b + c)! (-a + b + c)! (a - \alpha)! (b - \beta)!} \right]^{1/2}$$

$\langle n' j' m' | \mathfrak{M}_{k\kappa} | n j m \rangle$  obeys the same set of recurrence relations as  $C_{j m k \kappa}^{j' m'}$ .

Therefore there is a direct proportionality between these values. There is the algebraic proof of the Wigner-Eckart theorem.

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Proof (2nd variant)

$$\langle n' j' m' | \mathfrak{M}_{k\kappa} | n j m \rangle \equiv \int dX' \langle n' j' m' | X' \rangle \mathfrak{M}_{k\kappa}(X') \langle X' | n j m \rangle$$

where  $\int dX'$  symbolizes integration over all angular variables and summation over all spin variables, as well as integration over all radial variables.

Now consider the co-ordinates  $X'$  as rotated with respect to  $X$  by a transformation given by the Euler angles  $\alpha, \beta, \gamma$ .

The Jacobian of the transformation from  $X'$  to  $X$  is 1.

$$\begin{aligned}
\langle n' j' m' | \mathfrak{M}_{k\kappa} | n j m \rangle &= \\
&= \sum_{m'_1, \kappa_1, m_1} \int dX \langle n' j' m'_1 | X \rangle \mathfrak{M}_{k\kappa_1}(X) \langle X | n j m_1 \rangle \times \\
&\quad \times D_{m'_1 m'}^{j' *}(\alpha, \beta, \gamma) D_{m_1 m}^j(\alpha, \beta, \gamma) D_{\kappa_1 \kappa}^k(\alpha, \beta, \gamma) = \\
&= \sum_{m'_1, \kappa_1, m_1} \langle n' j' m'_1 | \mathfrak{M}_{k\kappa_1} | n j m_1 \rangle D_{m'_1 m'}^{j' *}(\alpha, \beta, \gamma) D_{m_1 m}^j(\alpha, \beta, \gamma) D_{\kappa_1 \kappa}^k(\alpha, \beta, \gamma)
\end{aligned}$$

But this rotation can be taken arbitrary. We can average over all possible rotations by integrating and dividing by the integration volume:

$$\begin{aligned}
\langle n' j' m' | \mathfrak{M}_{k\kappa} | n j m \rangle &= \sum_{m'_1, \kappa_1, m_1} \langle n' j' m'_1 | \mathfrak{M}_{k\kappa_1} | n j m_1 \rangle \times \\
&\times \frac{1}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^\pi \sin \beta d\beta \int_0^{2\pi} d\gamma D_{m'_1 m'}^{j' *}(\alpha, \beta, \gamma) D_{m_1 m}^j(\alpha, \beta, \gamma) D_{\kappa_1 \kappa}^k(\alpha, \beta, \gamma)
\end{aligned}$$

Recall that

$$\begin{aligned}
\int_0^{2\pi} d\alpha \int_0^\pi d\beta \sin \beta \int_0^{2\pi} d\gamma D_{M_3 M'_3}^{J_3 *}( \alpha, \beta, \gamma ) D_{M_2 M'_2}^{J_2}( \alpha, \beta, \gamma ) D_{M_1 M'_1}^{J_1}( \alpha, \beta, \gamma ) &= \\
&= \frac{8\pi^2}{2J_3 + 1} C_{J_1 M_1 J_2 M_2}^{J_3 M_3} C_{J_1 M'_1 J_2 M'_2}^{J_3 M'_3}
\end{aligned}$$

$$\langle n' j' m' | \mathfrak{M}_{k\kappa} | n j m \rangle = \sum_{m'_1, \kappa_1, m_1} \langle n' j' m'_1 | \mathfrak{M}_{k\kappa_1} | n j m_1 \rangle \frac{1}{2j' + 1} C_{jm_1 k \kappa_1}^{j' m'_1} C_{jm k \kappa}^{j' m'}$$

Substitution of  $\langle n' j' m' | \mathfrak{M}_{k\kappa} | n j m \rangle = (-1)^{2k} C_{jm k \kappa}^{j' m'} \frac{\langle n' j' || \mathfrak{M}_{k\kappa} || n j \rangle}{\sqrt{2j' + 1}}$

satisfies this equality identically.

Note that  $\sum_{\kappa_1, m_1} \left( C_{jm_1 k \kappa_1}^{j' m'_1} \right)^2 = 1$

and the subsequent summation over  $m'_1$  removes the prefactor  $\frac{1}{2j'+1}$ .

Sum rules  $\sum_{m_x} |\langle n' j' m' | \hat{\mathfrak{M}}_{kx} | n j m \rangle|^2 = \frac{|\langle n' j' || \hat{\mathfrak{M}}_k || n j \rangle|^2}{2j' + 1},$

$$\sum_{m' x} |\langle n' j' m' | \hat{\mathfrak{M}}_{kx} | n j m \rangle|^2 = \frac{|\langle n' j' || \hat{\mathfrak{M}}_k || n j \rangle|^2}{2j + 1},$$

$$\sum_{m m'} |\langle n' j' m' | \hat{\mathfrak{M}}_{kx} | n j m \rangle|^2 = \frac{|\langle n' j' || \hat{\mathfrak{M}}_k || n j \rangle|^2}{2k + 1},$$

$$\sum_{m m' x} |\langle n' j' m' | \hat{\mathfrak{M}}_{kx} | n j m \rangle|^2 = |\langle n' j' || \hat{\mathfrak{M}}_k || n j \rangle|^2.$$

follow from the Wigner-Eckart theorem and the normalization of  $3jm$ -symbols.

# V. Wigner 6j-symbols

Addition of three angular momenta can be realized using three schemes:

$$\begin{aligned} \text{I) } & \mathbf{j}_1 + \mathbf{j}_2 = \mathbf{j}_{12}, & \mathbf{j}_{12} + \mathbf{j}_3 = \mathbf{j}, \\ \text{II) } & \mathbf{j}_2 + \mathbf{j}_3 = \mathbf{j}_{23}, & \mathbf{j}_1 + \mathbf{j}_{23} = \mathbf{j}, \\ \text{III) } & \mathbf{j}_1 + \mathbf{j}_3 = \mathbf{j}_{13}, & \mathbf{j}_{13} + \mathbf{j}_2 = \mathbf{j}. \end{aligned}$$

I) Eigenstates of  $\hat{j}_1^2, \hat{j}_2^2, \hat{j}_3^2, \hat{j}_{12}^2, \hat{j}^2, \hat{j}_z$

$$|j_1 j_2 (j_{12}) j_3 j m\rangle = \sum_{m_1 m_2 m_3} C_{j_{12} m_{12} j_3 m_3}^{j m} C_{j_1 m_1 j_2 m_2}^{j_{12} m_{12}} |j_1 m_1, j_2 m_2, j_3 m_3\rangle$$

II) Eigenstates of  $\hat{j}_1^2, \hat{j}_2^2, \hat{j}_3^2, \hat{j}_{23}^2, \hat{j}^2, \hat{j}_z$

$$|j_1, j_2 j_3 (j_{23}) j m\rangle = \sum_{m_1 m_2 m_3} C_{j_1 m_1 j_{23} m_{23}}^{j m} C_{j_2 m_2 j_3 m_3}^{j_{23} m_{23}} |j_1 m_1, j_2 m_2, j_3 m_3\rangle$$

I) Eigenstates of  $\hat{j}_1^2, \hat{j}_2^2, \hat{j}_3^2, \hat{j}_{13}^2, \hat{j}^2, \hat{j}_z$

$$|j_1 j_3 (j_{13}) j_2 j m\rangle = \sum_{m_1 m_2 m_3} C_{j_{13} m_{13} j_2 m_2}^{j m} C_{j_1 m_1 j_3 m_3}^{j_{13} m_{13}} |j_1 m_1, j_2 m_2, j_3 m_3\rangle$$

The unitary transformation between the basis functions corresponding to different coupling schemes:

$$\begin{aligned} \langle j_1 j_2 (j_{12}) j_3 j m | j_1, j_2 j_3 (j_{23}) j' m' \rangle &= \delta_{j j'} \delta_{m m'} U(j_1 j_2 j_3; j_{12} j_{23}) = \\ &= \delta_{j j'} \delta_{m m'} (-1)^{j_1 + j_2 + j_3 + j} \sqrt{(2j_{12} + 1)(2j_{23} + 1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}. \end{aligned}$$

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \quad \text{-- Wigner 6j-symbol}$$

Why the dependence of the transformation coefficients on the total momentum projections is reduced to the Kronecker delta-symbol?

This is due to the Wigner-Eckart theorem:

$$\begin{aligned} \langle j_1 j_2 (j_{12}) j_3 j m | j_1, j_2 j_3 (j_{23}) j' m' \rangle &= \langle j_1 j_2 (j_{12}) j_3 j m | \hat{I}_{00} | j_1, j_2 j_3 (j_{23}) j' m' \rangle \\ \langle j_1 j_2 (j_{12}) j_3 j m | \hat{I}_{00} | j_1, j_2 j_3 (j_{23}) j' m' \rangle &\propto C_{j' m' 00}^{j m} = \delta_{j' j} \delta_{m' m} \end{aligned}$$

Identity operator of the rank 0.

It follows from the definition:

$$\begin{aligned} & \sum C_{j_{12} m_{12} j_3 m_3}^{j m} C_{j_1 m_1 j_2 m_2}^{j_{12} m_{12}} C_{j_1 m_1 j_{23} m_{23}}^{j' m'} C_{j_2 m_2 j_3 m_3}^{j_{23} m_{23}} = \\ & = \delta_{j j'} \delta_{m m'} (-1)^{j_1 + j_2 + j_3 + j} \sqrt{(2j_{12} + 1)(2j_{23} + 1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \end{aligned}$$

The summation here is over  $m_1, m_2, m_3, m_{12}, m_{23}$ , while the values of  $m$  and  $m'$  are fixed.

For the two other coupling schemes we have

$$\begin{aligned} \langle j_1 j_2 (j_{12}) j_3 j m | j_1 j_3 (j_{13}) j_2 j' m' \rangle & = \delta_{j j'} \delta_{m m'} (-1)^{j + j_1 - j_{12} - j_{13}} U(j_2 j_1 j j_3; j_{12} j_{13}) = \\ & = \delta_{j j'} \delta_{m m'} (-1)^{j_2 + j_3 + j_{12} + j_{13}} \sqrt{(2j_{12} + 1)(2j_{13} + 1)} \begin{Bmatrix} j_2 & j_1 & j_{12} \\ j_3 & j & j_{13} \end{Bmatrix}, \end{aligned}$$

$$\begin{aligned} \langle j_1, j_2 j_3 (j_{23}) j m | j_1 j_3 (j_{13}) j_2 j' m' \rangle & = \delta_{j j'} \delta_{m m'} (-1)^{j_2 + j_3 - j_{23}} U(j_1 j_3 j j_2; j_{13} j_{23}) = \\ & = \delta_{j j'} \delta_{m m'} (-1)^{j_1 + j + j_{23}} \sqrt{(2j_{13} + 1)(2j_{23} + 1)} \begin{Bmatrix} j_1 & j_3 & j_{13} \\ j_2 & j & j_{23} \end{Bmatrix}. \end{aligned}$$



All the momenta in  $\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$  must be integer or half-integer.

The groups  $(j_1 j_2 j_{12})$ ,  $(j_{12} j_3 j)$ ,  $(j_2 j_3 j_{23})$  and  $(j_{23} j_1 j)$  must obey **the triangle rule**.

From the unitarity of the transformation we obtain the orthogonality and normalization conditions:

$$\sum_{j_{12}} (2j_{12} + 1) (2j_{23} + 1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j'_{23} \end{Bmatrix} = \delta_{j_{23} j'_{23}},$$

$$\sum_{j_{23}} (2j_{12} + 1) (2j_{23} + 1) \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j'_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \delta_{j_{12} j'_{12}}.$$

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \equiv (-1)^{a+b+d+e} W(abed; cf)$$

Racah coefficient

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \sum (-1)^{d+e+f+\delta+\varepsilon+\varphi} \begin{Bmatrix} a & b & c \\ \alpha & \beta & \gamma \end{Bmatrix} \begin{Bmatrix} a & e & f \\ \alpha & \varepsilon & -\varphi \end{Bmatrix} \begin{Bmatrix} d & b & f \\ -\delta & \beta & \varphi \end{Bmatrix} \begin{Bmatrix} d & e & c \\ \delta & -\varepsilon & \gamma \end{Bmatrix}$$

The sum here is formally over  $\alpha, \beta, \gamma, \delta, \varepsilon, \varphi$ , but only three summation indices are independent, since the sum of the numbers in the lower row of a  $3jm$ -symbol must be zero.

From this expression we can conclude that the value of a  $6j$ -symbols remains the same after any transposition of its columns as well as after simultaneous „flipping“ of any two columns (such that the upper and lower elements change their places):

$$\begin{aligned} & \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Bmatrix} a & c & b \\ d & f & e \end{Bmatrix} = \begin{Bmatrix} b & a & c \\ e & d & f \end{Bmatrix} = \begin{Bmatrix} b & c & a \\ e & f & d \end{Bmatrix} = \begin{Bmatrix} c & a & b \\ f & d & e \end{Bmatrix} = \begin{Bmatrix} c & b & a \\ f & e & d \end{Bmatrix} = \\ & = \begin{Bmatrix} a & e & f \\ d & b & c \end{Bmatrix} = \begin{Bmatrix} a & f & e \\ d & c & b \end{Bmatrix} = \begin{Bmatrix} e & a & f \\ b & d & c \end{Bmatrix} = \begin{Bmatrix} e & f & a \\ b & c & d \end{Bmatrix} = \begin{Bmatrix} f & a & e \\ c & d & b \end{Bmatrix} = \begin{Bmatrix} f & e & a \\ c & b & d \end{Bmatrix} = \\ & = \begin{Bmatrix} d & e & c \\ a & b & f \end{Bmatrix} = \begin{Bmatrix} d & c & e \\ a & f & b \end{Bmatrix} = \begin{Bmatrix} e & d & c \\ b & a & f \end{Bmatrix} = \begin{Bmatrix} e & c & d \\ b & f & a \end{Bmatrix} = \begin{Bmatrix} c & d & e \\ f & a & b \end{Bmatrix} = \begin{Bmatrix} c & e & d \\ f & b & a \end{Bmatrix} = \\ & = \begin{Bmatrix} d & b & f \\ a & e & c \end{Bmatrix} = \begin{Bmatrix} d & f & b \\ a & c & e \end{Bmatrix} = \begin{Bmatrix} b & d & f \\ e & a & c \end{Bmatrix} = \begin{Bmatrix} b & f & d \\ e & c & a \end{Bmatrix} = \begin{Bmatrix} f & d & b \\ c & a & e \end{Bmatrix} = \begin{Bmatrix} f & b & d \\ c & e & a \end{Bmatrix}. \end{aligned}$$

Regge symmetry properties:

$$\begin{aligned} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} &= \begin{Bmatrix} a & s_1 - b & s_1 - c \\ d & s_1 - e & s_1 - f \end{Bmatrix} = \begin{Bmatrix} s_2 - a & b & s_2 - c \\ s_2 - d & e & s_2 - f \end{Bmatrix} = \\ &= \begin{Bmatrix} s_3 - a & s_3 - b & c \\ s_3 - d & s_3 - e & f \end{Bmatrix} = \begin{Bmatrix} s_2 - d & s_3 - e & s_1 - f \\ s_2 - a & s_3 - b & s_1 - c \end{Bmatrix} = \begin{Bmatrix} s_3 - d & s_1 - e & s_2 - f \\ s_3 - a & s_1 - b & s_2 - c \end{Bmatrix}, \end{aligned}$$

where

$$s_1 = \frac{1}{2}(b + c + e + f), \quad s_2 = \frac{1}{2}(a + c + d + f), \quad s_3 = \frac{1}{2}(a + b + d + e).$$

If one of the indices = 0, then

$$\begin{aligned} \begin{Bmatrix} 0 & b & c \\ d & e & f \end{Bmatrix} &= (-1)^{b+e+d} \frac{\delta_{bc} \delta_{ef}}{\sqrt{(2b+1)(2e+1)}}, & \begin{Bmatrix} a & b & c \\ 0 & e & f \end{Bmatrix} &= (-1)^{a+b+e} \frac{\delta_{bf} \delta_{ce}}{\sqrt{(2b+1)(2c+1)}}, \\ \begin{Bmatrix} a & 0 & c \\ d & e & f \end{Bmatrix} &= (-1)^{a+d+e} \frac{\delta_{ac} \delta_{df}}{\sqrt{(2a+1)(2d+1)}}, & \begin{Bmatrix} a & b & c \\ d & 0 & f \end{Bmatrix} &= (-1)^{a+b+d} \frac{\delta_{af} \delta_{cd}}{\sqrt{(2a+1)(2c+1)}}, \\ \begin{Bmatrix} a & b & 0 \\ d & e & f \end{Bmatrix} &= (-1)^{a+e+f} \frac{\delta_{ab} \delta_{de}}{\sqrt{(2a+1)(2d+1)}}, & \begin{Bmatrix} a & b & c \\ d & e & 0 \end{Bmatrix} &= (-1)^{a+b+c} \frac{\delta_{ae} \delta_{bd}}{\sqrt{(2a+1)(2b+1)}}. \end{aligned}$$

$$\begin{Bmatrix} a & b & c \\ 1/2 & e & f \end{Bmatrix}$$

$f$	$e = c + 1/2$	$e = c - 1/2$
$b + 1/2$	$(-1)^{s+1} \frac{1}{2} \left[ \frac{(s+2)(s-2a+1)}{(2b+1)(b+1)(2c+1)(c+1)} \right]^{1/2}$	$(-1)^s \frac{1}{2} \left[ \frac{(s-2c+1)(s-2b)}{(2b+1)(b+1)c(2c+1)} \right]^{1/2}$
$b - 1/2$	$(-1)^s \frac{1}{2} \left[ \frac{(s-2c)(s-2b+1)}{b(2b+1)(2c+1)(c+1)} \right]^{1/2}$	$(-1)^s \frac{1}{2} \left[ \frac{(s+1)(s-2a)}{b(2b+1)c(2c+1)} \right]^{1/2}$

$$s = a + b + c,$$

$$\begin{Bmatrix} a & b & c \\ 1 & e & f \end{Bmatrix}$$

$f$	$e = c + 1$
$b + 1$	$(-1)^s \frac{1}{2} \left[ \frac{(s+2)(s+3)(s-2a+1)(s-2a+2)}{(2b+1)(b+1)(2b+3)(2c+1)(c+1)(2c+3)} \right]^{1/2}$
$b$	$(-1)^{s+1} \frac{1}{2} \left[ \frac{(s+2)(s-2c)(s-2b+1)(s-2a+1)}{b(2b+1)(b+1)(2c+1)(c+1)(2c+3)} \right]^{1/2}$
$b - 1$	$(-1)^s \frac{1}{2} \left[ \frac{(s-2c-1)(s-2c)(s-2b+1)(s-2b+2)}{(2b-1)b(2b+1)(2c+1)(c+1)(2c+3)} \right]^{1/2}$

$$\begin{Bmatrix} a & b & c \\ 1 & e & f \end{Bmatrix}$$

(continued)

$f$	$e = c$
$b + 1$	$(-1)^{s+1} \frac{1}{2} \left[ \frac{(s+2)(s-2c+1)(s-2b)(s-2a+1)}{(2b+1)(b+1)(2b+3)c(2c+1)(c+1)} \right]^{1/2}$
$b$	$(-1)^{s+1} \frac{1}{2} \frac{X}{[b(2b+1)(b+1)c(2c+1)(c+1)]^{1/2}}$
$b - 1$	$(-1)^s \frac{1}{2} \left[ \frac{(s+1)(s-2c)(s-2b+1)(s-2a)}{(2b-1)b(2b+1)c(2c+1)(c+1)} \right]^{1/2}$
$f$	$e = c - 1$
$b + 1$	$(-1)^s \frac{1}{2} \left[ \frac{(s-2c+1)(s-2c+2)(s-2b-1)(s-2b)}{(2b+1)(b+1)(2b+3)(2c-1)c(2c+1)} \right]^{1/2}$
$b$	$(-1)^s \frac{1}{2} \left[ \frac{(s+1)(s-2c+1)(s-2b)(s-2a)}{b(2b+1)(b+1)(2c-1)c(2c+1)} \right]^{1/2}$
$b - 1$	$(-1)^s \frac{1}{2} \left[ \frac{s(s+1)(s-2a-1)(s-2a)}{(2b-1)b(2b+1)(2c-1)c(2c+1)} \right]^{1/2}$

$$X = -a(a+1) + b(b+1) + c(c+1)$$