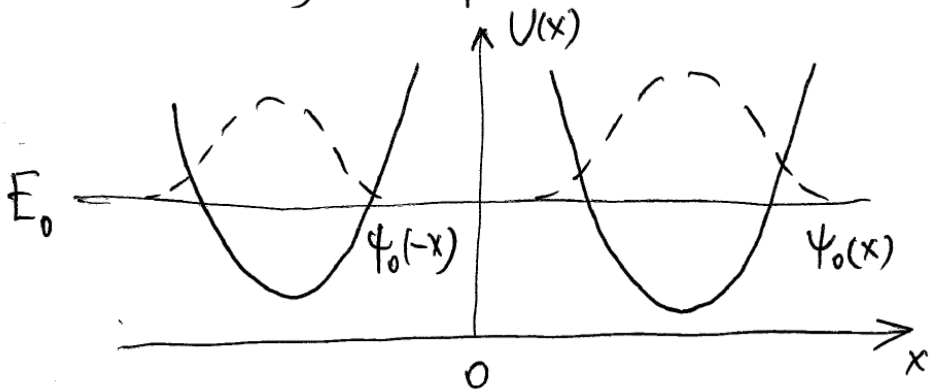
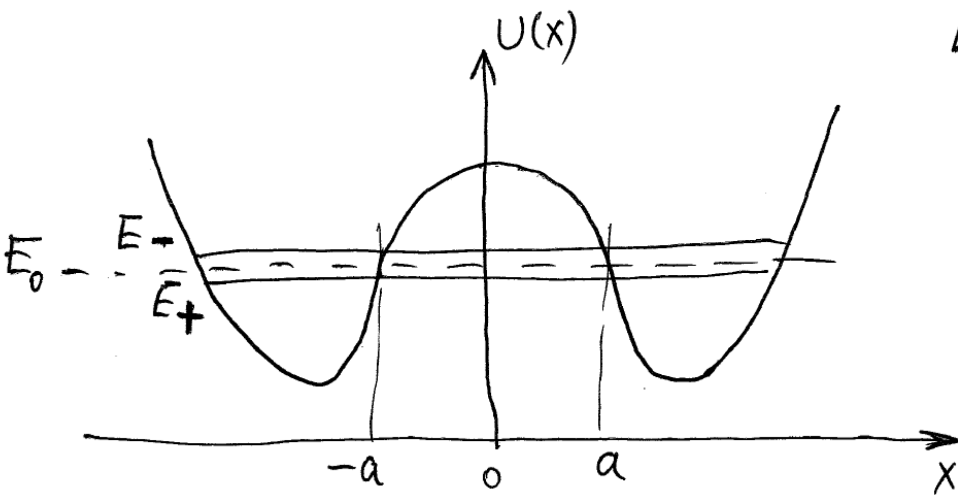
Tunneling phenomena in BEC

1. Single-particle tunneling



Infinite barrier: Two degenerate states with the Energy E_0 and wave functions $\psi_0(x)$ and $\psi_0(-x)$



$\pm a$ – classical turning points

Lower barrier: degeneracy is lifted, two states

$$\psi_\pm(x) \simeq \frac{\psi_0(x) \pm \psi_0(-x)}{\sqrt{2}}$$

have the energies

$$\frac{E_+ + E_-}{2} = E_0$$

$$E_- - E_+ = \frac{\hbar\omega}{\pi} \exp\left[-\int_{-a}^{a} |p|dx/\hbar\right]$$

ω = freq. of classical oscillations in a single well

2. Two-mode Hamiltonian for a BEC in a double-well trap ($E_0 \equiv 0$)

a) Hamiltonian of two isolated symmetric traps

$$\hat{H}_a = \frac{E_c}{2}\left(\hat{N}_L^2 + \hat{N}_R^2\right) \qquad \hat{N}_{L,R} = \hat{a}^+_{L,R}\hat{a}_{L,R}$$

$$E_c = \frac{4\pi\hbar^2 a}{m}\int d^3\vec{r}\, |\phi_0(\vec{r})|^4$$

(weak interactions, taken into account perturbatively)

$$\hat{H}_a |N_L, N_R\rangle = \frac{E_c}{2}(N_L^2 + N_R^2)\, |N_L, N_R\rangle$$

b) Hamiltonian of non-interacting bosons in a double-well trap

$$\hat{H}_b = -\hbar J\left(\hat{a}_L^+\hat{a}_R + \hat{a}_R^+\hat{a}_L\right)$$

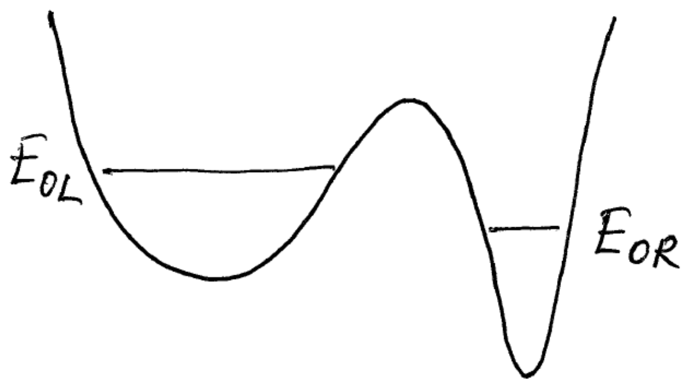
$$\hat{a}_\pm = \frac{1}{\sqrt{2}}\left(\hat{a}_L \pm \hat{a}_R\right)$$

$$\hat{H}_b = \hbar J\left(\hat{a}_-^+\hat{a}_- - \hat{a}_+^+\hat{a}_+\right) \qquad 2\hbar J = E_- - E_+$$

c) Combined Hamiltonian $\hat{H} = \hat{H}_a + \hat{H}_b$

$$\hat{H} = \frac{E_{CL}}{2}(\hat{a}_L^+ \hat{a}_L)^2 + \frac{E_{CR}}{2}(\hat{a}_R^+ \hat{a}_R) - \hbar\gamma(\hat{a}_L^+ \hat{a}_R + \hat{a}_R^+ \hat{a}_L) +$$

$$+ \frac{1}{2}(E_{0L} - E_{0R})(\hat{a}_L^+ \hat{a}_L - \hat{a}_R^+ \hat{a}_R)$$



Asymmetric trap.

$$i\hbar \frac{\partial}{\partial t}\hat{a}_L = E_{CL}\hat{a}_L^+ \hat{a}_L \hat{a}_L + \frac{1}{2}(E_{0L} - E_{0R})\hat{a}_L - \hbar\gamma \hat{a}_R$$

$$i\hbar \frac{\partial}{\partial t}\hat{a}_R = E_{CR}\hat{a}_R^+ \hat{a}_R \hat{a}_R - \frac{1}{2}(E_{0L} - E_{0R})\hat{a}_R - \hbar\gamma a_L$$

$$N_T \equiv \hat{N}_L + \hat{N}_R = \hat{a}_L^+ \hat{a}_L + \hat{a}_R^+ \hat{a}_R = \text{const}$$

(integral of motion)

4. Mean-field treatment

A. Smerzi, S. Fantoni, S. Giovanazzi, and S.R. Shenoy
Phys. Rev. Lett. **79**, 4950 (1997)

$$\hat{a}_{L,R} \to \sqrt{N_{L,R}}\, e^{i\theta_{L,R}}$$

$$z = \frac{N_L - N_R}{N_L + N_R} \qquad \phi = \theta_R - \theta_L \qquad -1 \leq z \leq 1$$

Dimensionless time $\tau = 2\mathcal{J} t$, $\dot{\phi} \equiv \frac{d\phi}{d\tau} = \frac{1}{2\mathcal{J}} \frac{d\phi}{dt}$

$$\dot{z} = -\sqrt{1-z^2}\, \sin\phi$$

$$\dot{\phi} = \Lambda z + \frac{z}{\sqrt{1-z^2}} \cos\phi + \Delta E$$

$$\Delta E = \frac{E_{0L} + E_{0R}}{2\hbar \mathcal{J}} + \frac{(E_{cL} - E_{cR})(N_L + N_R)}{4\hbar \mathcal{J}}$$

$$\Lambda = \frac{(E_{cL} + E_{cR})(N_L + N_R)}{4\hbar \mathcal{J}}$$

We consider here repulsive interactions, $E_c > 0$, otherwise the system may be unstable

The classical variables z and ϕ are canonically conjugate

$$\dot{z} = -\frac{\partial H_{cl}}{\partial \phi} \qquad \dot{\phi} = \frac{\partial H_{cl}}{\partial z}$$

$$H_{cl} = \frac{\Lambda}{2} z^2 - \sqrt{1-z^2}\cos\phi + \Delta E\, z$$

Different regimes of oscillations

a) far-detuned ground states of two wells $|E_{0L}-E_{0R}| \gg E_{CL,R} N_{L,R}, \hbar J$

$$z \approx \text{const} \qquad \phi \approx \frac{1}{\hbar}(E_{0L}-E_{0R})t$$

(M. Saba et al., Science 307, 1945 (2005) experiment)

b) From now on, set $E_{0L} \equiv E_{0R}$ and $E_{CL} \equiv E_{CR} = E_C$

b1) Non-interacting limit, $\Lambda \to 0$

$$z(t) = z_{max}\cos(2Jt+\alpha_0) \qquad \phi(t) = z_{max}\sin(2Jt+\alpha_0)$$

(oscillations with the single-particle tunneling frequency $2J$)

b2) Josephson oscillation regime $|z| \ll 1$ $|\phi| \ll 1$

$$\dot{z} = -\phi \qquad \dot{\phi} = (\Lambda + 1) z$$

Oscillations at the frequency $\sqrt{\Lambda+1}$, or, in dimensional units, $\omega_0 = \sqrt{2J(2J + E_C(N_L+N_R)}/\hbar$

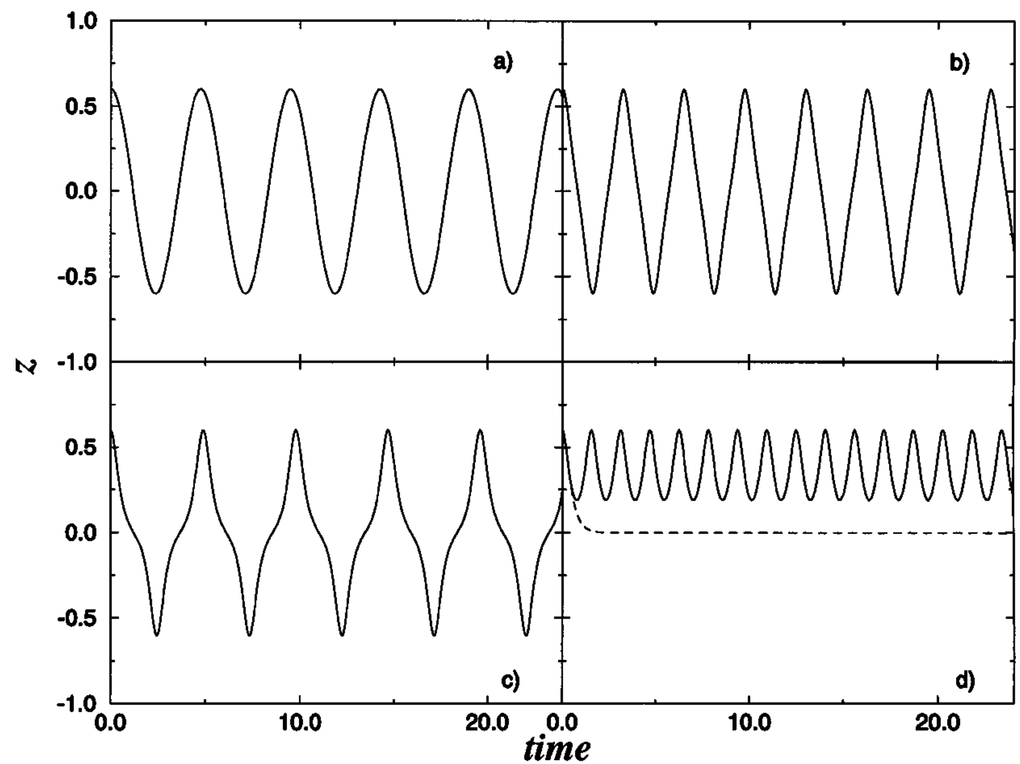
b3) Nonlinear regime

$$\Lambda > \Lambda_{cr} = 2\left(\sqrt{1-z^2(0)}\cos\phi(0) + 1\right)/z^2(0) \Longrightarrow$$

$\Longrightarrow$ Macroscopic population self-trapping

$z(t)$ for $z(0) = 0,6$ $\phi(0) = 0$

$\Lambda = 1$ (a), $= 8$ (b), $= 9,99$ (c), $= \Lambda_{cr} = 10$ (dashed line, d), $= 11$ (solid line, d).



5. Quantum treatment:

G. S. Paraoanu, S. Kohler, F. Sols, and A. J. Leggett

J. Phys. B **34**, 4689 (2001)

Assume: atoms are condensed in the lowest-energy mode

$$\hat{c}_0 = \frac{1}{\sqrt{2}}(\hat{a}_L + \hat{a}_R) \equiv \hat{a}_+$$

The mode $\hat{c}_1 = \frac{1}{\sqrt{2}}(\hat{a}_L - \hat{a}_R) \equiv \hat{a}_-$ is treated as $\hat{c}_1 = u\hat{\gamma} - v\hat{\gamma}^+$ (Bogoliubov transformation) and yields the <u>quantum depletion</u>. Bogoliubov excit. annihilation operator $\hat{\gamma}$

The standard expression for the Josephson oscillation frequency $\omega_J = \sqrt{2J(2J + E_c N_T/\hbar)}$ is retrieved

$$N_{12} = \frac{N_L - N_R}{2} \qquad \phi = \theta_R - \theta_L$$

In equilibrium: $\langle N_L \rangle = \langle N_R \rangle \equiv N_T/2$

$$N_T = N_L + N_R = \text{const}$$

Ground-state fluctuations:

$$\Delta N_{12} = \frac{\sqrt{N_T}}{2}\left(\frac{2\hbar\mathcal{J}}{E_C N_T + 2\hbar\mathcal{J}}\right)^{1/4} \qquad \Delta\phi = \frac{1}{\sqrt{N_T}}\left(\frac{E_C N_T + 2\hbar\mathcal{J}}{2\hbar\mathcal{J}}\right)^{1/4}$$

i) Non-interacting limit: $E_C N_T \ll \hbar\mathcal{J}$

$$\Delta N_{12} \approx \frac{\sqrt{N_T}}{2} \qquad \Delta\phi \simeq \frac{1}{\sqrt{N_T}} \qquad \omega_{\mathcal{J}} \simeq 2\mathcal{J}$$

ii) Intermediate interaction strength: $\frac{\hbar\mathcal{J}}{N_T} \lesssim E_C \lesssim N_T\hbar\mathcal{J}$

$$1 \lesssim \Delta N_{12} \lesssim \frac{\sqrt{N_T}}{2} \qquad \frac{1}{\sqrt{N_T}} \lesssim \Delta\phi \lesssim 1$$ and always $\Delta N_{12}\Delta\phi = \frac{1}{2}$

Squeezing!

$$\omega_{\mathcal{J}} \simeq \sqrt{2\mathcal{J}E_C N_T/\hbar}$$

iii) Very strong interactions: $E_C \gg \hbar\mathcal{J}N_T$

$\Delta N_{12} \ll 1$, $\Delta\phi \gg 1$ (phase is not determined), $\langle \hat{c}_1^{+}\hat{c}_1\rangle \sim \frac{N_T}{2}$

$\Rightarrow$ Fragmented state $|N_L, N_R\rangle$ instead of condensation in the "+" mode.

6. Coupled 1D quasicondensates

$$\hat{H} = \sum_{\alpha=L,R} \int dz \left(\frac{\hbar^2}{2m} \frac{\partial \hat{\psi}_\alpha^+}{\partial z} \frac{\partial \hat{\psi}_\alpha}{\partial z} + g_{1D} \hat{\psi}_\alpha^+ \hat{\psi}_\alpha^+ \hat{\psi}_\alpha \hat{\psi}_\alpha \right) - \hbar \mathcal{J} \int dz \left(\hat{\psi}_L^+ \hat{\psi}_R + \hat{\psi}_R^+ \hat{\psi}_L \right)$$

N. K. Whitlock and I. Bouchoule, Phys. Rev. A 68, 053609 (2003)

$$\hat{\psi}_\alpha = \sqrt{\bar{n}_{1D} + \delta\hat{n}_\alpha}\, e^{i\hat{\theta}_\alpha} \qquad \alpha = L, R$$

Symmetric ("charge", "in-phase") mode

$$\delta\hat{n}_+ = \delta\hat{n}_L + \delta\hat{n}_R \qquad \hat{\phi}_+ = \frac{1}{2}(\hat{\theta}_L + \hat{\theta}_R)$$

$$\omega_+(k) = \sqrt{\frac{\hbar k^2}{2m}\left(\frac{\hbar k^2}{2m} + 2 g_{1D} \bar{n}_{1D}/\hbar\right)}, \qquad S_+(k) = \frac{\hbar k^2}{2m\, \omega_+(k)}$$

Antisymmetric ("spin", "out-of-phase") mode

$$\omega_-(k) = \sqrt{\left(\frac{\hbar k^2}{2m} + 2\mathcal{J}\right)\left(\frac{\hbar k^2}{2m} + 2\mathcal{J} + 2 g_{1D} \bar{n}_{1D}/\hbar\right)}, \qquad S_-(k) = \frac{\frac{\hbar k^2}{2m} + 2\mathcal{J}}{\omega_-(k)}$$