Dynamics of a BEC at $T = 0$ ($T \ll T_c$)

I. Variational approach

I a) Ground State

$$\mu \Psi_0 = -\frac{\hbar^2}{2m} \nabla^2 \Psi_0 + V_{ext}(\vec{r}) \Psi_0 + g |\Psi_0|^2 \Psi_0$$

Non-interacting gas $\quad g = 0$

$$V_{ext}(\vec{r}) = \frac{m \omega_0^2 r^2}{2}$$

$$\mu = \frac{3}{2} \hbar \omega_0 , \qquad \Psi_0^{ni} = \frac{\sqrt{N}}{\left(\sqrt{\sqrt{\pi}\, l_0}\right)^3} e^{-\frac{1}{2}\frac{r^2}{l_0^2}} , \qquad l_0 = \sqrt{\frac{\hbar}{m\omega_0}}$$

Assume that for interacting gas ($g \neq 0$)

$$\Psi_0 = \frac{\sqrt{N}}{\left(\sqrt{\sqrt{\pi}\, \sigma}\right)^3} e^{-\frac{1}{2}\frac{r^2}{\sigma^2}} \qquad \int d^3\vec{r}\, |\Psi_0|^2 = N$$

and minimize $\mathcal{E}_\sigma = \int d^3\vec{r} \left(\frac{\hbar^2}{2m} |\nabla \Psi_0|^2 + \frac{m\omega_0^2}{2} r^2 |\Psi_0|^2 + \frac{g}{2} |\Psi_0|^4 \right)$

$$\frac{1}{N}\mathcal{E}_\sigma = \frac{3}{4}\frac{\hbar^2}{m\sigma^2} + \frac{3}{4}m\omega_0^2\sigma^2 + \frac{gN}{2(2\pi)^{3/2}\sigma^3}$$

$$g = \frac{4\pi\hbar^2}{m}a_s \qquad\qquad \frac{\partial \mathcal{E}_\sigma}{\partial\sigma} = 0 \implies \sigma$$

<u>$a_s > 0$</u>

$$N a_s / l_0 \ll 1$$

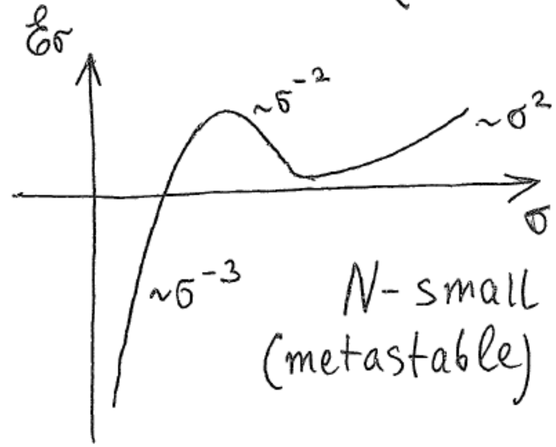
$$\sigma \simeq l_0\left(1 + \frac{2N}{\sqrt{2\pi}}\frac{a_s}{l_0}\right)$$

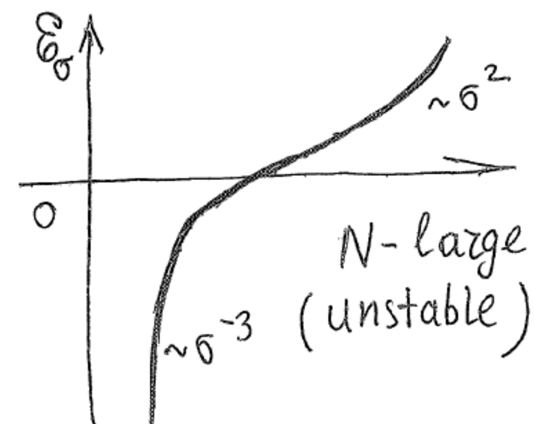
$$N a_s / l_0 \gg 1$$

$$\sigma \approx l_0\left(\sqrt{\frac{2}{\pi}}\,N\frac{a_s}{l_0}\right)^{1/5}$$

$$N_{cr} \sim \frac{l_0}{|a_s|}$$

<u>$a_s < 0$</u>





IIa) Dynamics

$$\mu \Psi_0 \rightarrow i\hbar \frac{\partial}{\partial t}\Psi_0$$

$$\mathcal{L}_{\sigma\beta} = \frac{i\hbar}{2}\int d^3\vec{r}\left(\Psi_0^* \frac{\partial}{\partial t}\Psi_0 - \frac{\partial \Psi_0^*}{\partial t}\Psi_0\right) - \mathcal{E}_\sigma$$

$$\Psi_0(\vec{r}) = \frac{\sqrt{N}}{(\sqrt{\pi}\,\sigma)^{3/2}} \exp\left[-\frac{1}{2}\frac{r^2}{\sigma^2} + \frac{i}{2}\beta r^2\right]$$

$\sigma = \sigma(t) \qquad \beta = \beta(t) \qquad \int d^3\vec{r}\,|\Psi_0|^2 = N$

Under harmonic isotropic confinement $\frac{m\omega_0^2}{2} r^2$

$$\frac{1}{N}\mathcal{L}_{\sigma\beta} = -\frac{3}{4}\hbar\dot{\beta}\sigma^2 - \frac{3}{4}\frac{\hbar^2}{m}\beta^2\sigma^2 - \frac{3}{4}\frac{\hbar^2}{m\sigma^2} - \frac{3}{4}m\omega_0^2\sigma^2 - \frac{gN}{4\pi\sqrt{2\pi}\,\sigma^3}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{\sigma\beta}}{\partial \dot{\beta}} = \frac{\partial}{\partial \beta}\mathcal{L}_{\sigma\beta} \qquad \beta = \frac{m}{\hbar\sigma}\dot{\sigma}$$

$$\frac{\partial}{\partial \sigma}\mathcal{L}_{\sigma\beta} = 0 \qquad \frac{3}{2}m\ddot{\sigma} = -\frac{\partial}{\partial \sigma}\left[\frac{3}{4}\frac{\hbar^2}{m\sigma^2} + \frac{3}{4}m\omega_0^2\sigma^2 + \frac{\hbar^2 N a_s}{m\sqrt{2\pi}\,\sigma^3}\right]$$

Linearization $\Rightarrow$ frequency of oscillations

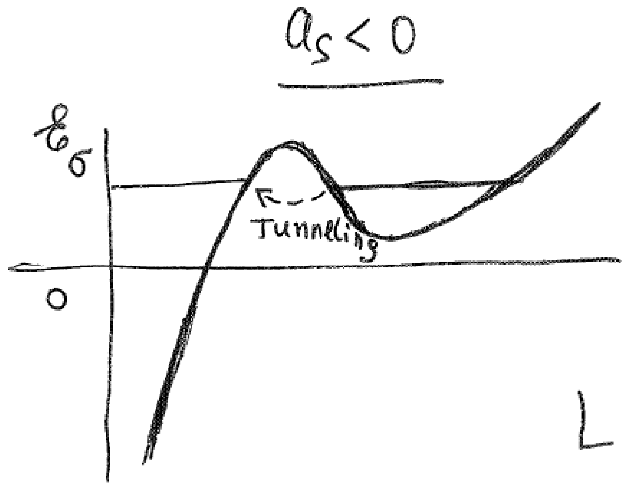
Monopole ($Na_s/\ell \ll 1$)

$$\omega_M \simeq 2\omega_0 \left(1 - 4\sqrt{\frac{Na_s}{l_0}} \right)$$

Dipole $\omega_D = \omega_0$

Quadrupole

$$\omega_Q = 2\omega_0 \left(1 + \mathcal{O}\left(\sqrt{\frac{Na_s}{l_0}} \right) \right)$$



$$m\ddot{\sigma} = F(\sigma) \equiv -\frac{\partial}{\partial\sigma}\Pi(\sigma)$$

We construct the respective Lagrangian

$$L = \frac{1}{2}m\dot{\sigma}^2 - \Pi(\sigma)$$

Canonical momentum $\quad p_\sigma = \frac{\partial L}{\partial\dot{\sigma}} = m\dot{\sigma}$

Hamiltonian $\quad H_\sigma = p_\sigma\dot{\sigma} - L = \frac{1}{2}m\dot{\sigma}^2 + \Pi(\sigma)$

$$H_\sigma = \frac{1}{2m}p_\sigma^2 + \Pi(\sigma)$$

Quantize this Hamiltonian: $\sigma \to \hat{\sigma}$, $p \to \hat{p}_\sigma$

$$[\hat{p}_\sigma, \hat{\sigma}] = -i\hbar$$

$i\hbar\frac{\partial}{\partial t}\chi(\sigma,t) = \hat{H}_\sigma\chi(\sigma,t) \Rightarrow$ Tunneling dynamics!

II. Quantum hydrodynamics

$$i\hbar \frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + U(\vec{r})\Psi + g|\Psi|^2\Psi$$

$$\Psi = \sqrt{n(\vec{r},t)}\, e^{i\varphi(\vec{r},t)}$$

Analogy with a quasiclassical (WKB) w. f. :

$$\Psi \sim \exp\left[\frac{i}{\hbar}\left(\int p_x dx + \int p_y dy + \int p_z dz\right)\right]$$

$$\vec{v} = \frac{\hbar \nabla\varphi}{m} \qquad \mathrm{rot}\,\vec{v} = 0$$

$$\frac{\partial}{\partial t} n + \nabla(n\vec{v}) = 0 \qquad \text{continuity}$$

$$m\frac{\partial \vec{v}}{\partial t} = -\nabla\left(\frac{m\vec{v}^2}{2} + U(\vec{r}) + gn - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right)$$

$$\mathrm{rot}\,\vec{v} = 0 \Rightarrow \nabla(v^2/2) = (\vec{v}\nabla)\vec{v} \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\nabla$$

$$m\frac{d}{dt}\vec{v} = -\nabla\left(U(\vec{r}) + gn - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right)$$

Uniform gas, $\quad V(\vec{r}) \equiv 0$

$n = n_0 + \delta n \quad , \quad \vec{v} = \delta\vec{v}$

$\delta n \sim e^{i\vec{k}\vec{r} - i\omega_k t} \qquad \delta\vec{v} \sim e^{i\vec{k}\vec{r} - i\omega_k t}$

$$\hbar\omega_k = \sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2gn_0\right)}$$

Steady-state: $\quad \vec{v} \equiv 0 \qquad \frac{\partial n}{\partial t} \equiv 0$

$$-\frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n} + V(\vec{r}) + gn = \mu \qquad \underline{\underline{\text{GPE}}}$$

Assume a spherically-symmetric trap $V(\vec{r}) = \frac{m\omega_{tr}^2}{2} r^2$ and neglect the „quantum pressure" $-\frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}$

$$\mu = \frac{1}{2} m\omega_{tr}^2 r^2 + gn \Rightarrow n(\vec{r}) = \begin{cases} n_0\left(1 - r^2/R^2\right) & , \; r<R \\ 0 & , \; r>R \end{cases}$$

Thomas-Fermi limit

$$n_0 = \mu/g \qquad R = \sqrt{\frac{2\mu}{m\omega_{tr}^2}}$$

$$N = 4\pi \int_0^R dr\, r^2 n_0 \left(1 - \frac{r^2}{R^2}\right) = \frac{8\pi}{15} n_0 R^3$$

$$R = l_0 \left(15 N \frac{a_s}{l_0}\right)^{1/5} \qquad N a_s / l_0 \gg 1$$

$$\mu = g n_0 \sim N^{2/5} \qquad {}^{87}Rb: \quad a_s = 5.3\ nm$$

The density profile differs from the TF profile in a thin surface layer where $\frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} \gtrsim g n$

Quantum hydrodynamical excitations

$$\frac{\partial}{\partial t} \vec{v} = -\frac{g}{m} \nabla \delta n \qquad \frac{\partial \delta n}{\partial t} = -\nabla (n \vec{v}) \ , \quad n = n_0 \left(1 - \frac{r^2}{R^2}\right)$$

$$-\frac{\partial^2}{\partial t^2} \delta n + \frac{g n_0}{m} \nabla \left[\left(1 - \frac{r^2}{R^2}\right) \nabla \delta n \right] = 0$$

δn - finite at $r \to 0$ and $r \to R$

$$\delta n \sim e^{-i\omega_{nl} t} \left(\frac{r}{R}\right)^l F_{nl}\left(\frac{r}{R}\right) e^{im\varphi} P_l^{|m|}(\cos\theta) \qquad \omega_{nl} = \omega_{tr} \sqrt{2n^2 + 2nl + 3n + l}$$

$$n = 0, 1, 2, 3, \ldots$$

Non-interacting gas $\quad \omega_{nl} = \omega_{tr} (2n + l)$

Coincidence for the dipole mode only $\omega_{n=0\ l=1} = \omega_{tr}$

Surface modes in the TF limit

$$\omega_{n=0\ l} = \omega_{tr} \sqrt{l}$$

Solitons

Dimensionality reduction in GPE

$$V_{ext}(\vec{r}) = \frac{m}{2}\left(\omega_{\parallel}^2 z^2 + \omega_{\perp}^2 (x^2+y^2)\right) \quad , \quad \omega_{\perp} \gg \omega_{\parallel}$$

The roughest approximation

$$\Psi(\vec{r},t) = \psi_{\perp 0}(x,y)\, \Phi(z,t),$$

with $\psi_{\perp 0}$ = ground state of the radial trapping Hamiltonian

$$\hat{H}_{\perp} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{m}{2}\omega_{\perp}^2 (x^2+y^2)$$

This holds if typical energy of the processes in z-direction is $\ll \hbar\omega_{\perp}$

1D GPE: $$i\hbar \frac{\partial}{\partial t}\Phi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + V_{1D}(z) + g_{1D}|\Phi|^2\right]\Phi$$

$$g_{1D} = \frac{4\pi\hbar^2 a}{m} \cdot \frac{1}{2\pi \ell_{\perp}^2} = 2\hbar\omega_{\perp} a \qquad \ell_{\perp} = \sqrt{\frac{\hbar}{m\omega_{\perp}}}$$

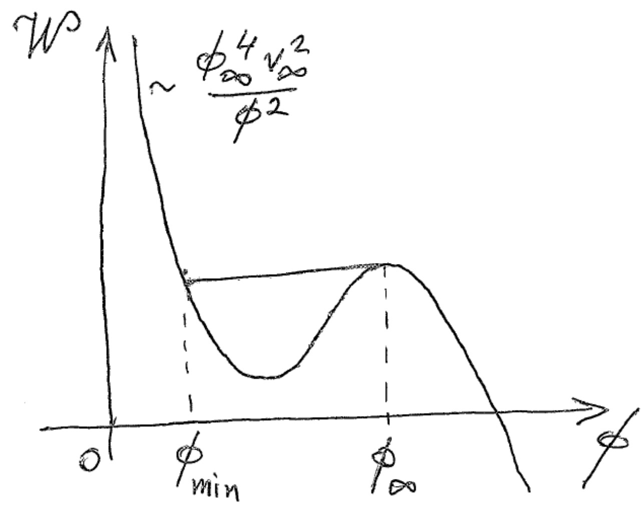
Quantum hydrodynamics in 1D (infinite tube, $V_{1D}(z) \equiv 0$)

$$\frac{\partial}{\partial t} n + \frac{\partial}{\partial z}(nv) = 0 \qquad m\frac{\partial}{\partial t} v = -\frac{\partial}{\partial z}\left(\frac{m}{2}v^2 + g_{1D}n - \frac{\hbar^2}{2m\sqrt{n}}\frac{\partial^2}{\partial z^2}\sqrt{n}\right)$$

Stationary solitonic solutions (the reference frame where a soliton is at rest): $\frac{\partial}{\partial t} n = 0 \qquad \frac{\partial}{\partial t} v = 0$

a) $\underline{g_{1D} > 0}$

$$v = \frac{n_\infty v_\infty}{n}$$



$$-\frac{\hbar^2}{2m\sqrt{n}}\frac{\partial^2}{\partial z^2}\sqrt{n} + g_{1D}n + \frac{m}{2}\frac{n_\infty^2 v_\infty^2}{n^2} = \mu$$

$z \to \infty \qquad \frac{\partial}{\partial z} n \to 0 \qquad \mu = g_{1D}n_\infty + \frac{m v_\infty^2}{2}$

Mechanical analogy:

z = "time" $\qquad \sqrt{n} \equiv \phi$ = "particle co-ordinate"

$$\phi'' = \frac{2mg_{1D}}{\hbar^2}\phi^3 + \frac{m^2}{\hbar^2}\frac{\phi_\infty^4 v_\infty^2}{\phi^3} - \frac{2m}{\hbar^2}\left(g_{1D}\phi_\infty^2 + \frac{m}{2}v_\infty^2\right)\phi$$

$$\frac{1}{2}\phi'^2 - \frac{mg_{1D}}{2\hbar^2}\phi^4 + \frac{m}{2\hbar^2}\frac{\phi_\infty^4 v_\infty^2}{\phi^2} + \frac{m}{\hbar^2}\left(g_{1D}\phi_\infty^2 + \frac{m}{2}v_\infty^2\right)\phi^2 = \underbrace{W}_{\text{"Energy"}} = \text{const}, \quad \phi'|_{\pm\infty} = 0$$

In the rest frame of the gas
(moving solitons)

$$\frac{u^2}{c^2} = \frac{n_{\min}}{n_\infty}$$

$$n(z,t) = n_{min} + (n_\infty - n_{min}) \tanh^2\left(\frac{z-ut}{\xi_u}\right)$$ Gray soliton

$$\xi_u = \frac{\hbar}{mc}\frac{1}{\sqrt{1-(u/c)^2}}, \quad \xi_0 = \frac{\hbar}{mc} = \text{healing length}, \quad c = \sqrt{\frac{g_{1D} n_\infty}{m}} = \text{speed of sound}$$

Dark soliton (extreme case):

$$n = n_\infty \tanh^2 z/\xi_0$$

Change of the phase throug the soliton

$$\Phi = |\Phi| e^{i\chi} \qquad \chi(z \to +\infty) - \chi(z \to -\infty) = -2\arccos\frac{n_{min}}{n_\infty}$$

б) $\underline{g_{1D} < 0}$ Bright soliton

$$\Phi(z,t) = \Phi(0)\, e^{-i\mu t/\hbar} \frac{1}{\cosh\left(\sqrt{2m|\mu|\hbar^{-2}}\, z\right)}$$

$$\mu = \frac{1}{2} g_{1D} |\Phi(0)|^2$$

Total number of atoms in the bright soliton

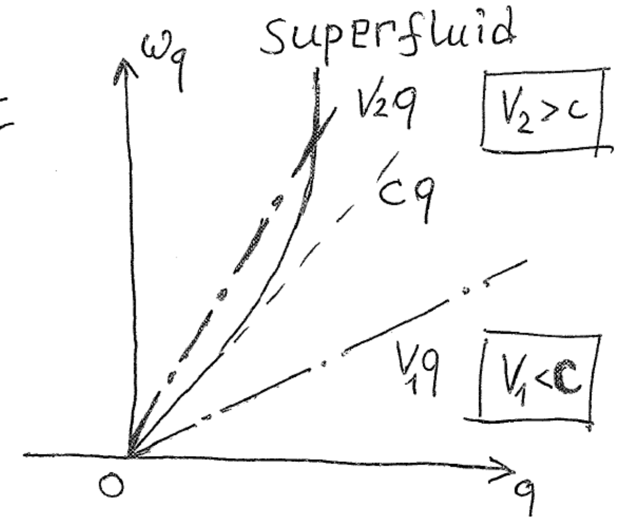
$$N_{bs} = \frac{2\hbar|\Phi(0)|}{\sqrt{m|g_{1D}|}}$$

Landau's criterion of superfluidity

A macroscopic perturbation (potential) moves through a fluid

$$\hat{H}_{int} = \int d^3\vec{r}\; U(\vec{r}-\vec{V}t)\, \hat{\psi}^+(\vec{r})\, \hat{\psi}(\vec{r})$$

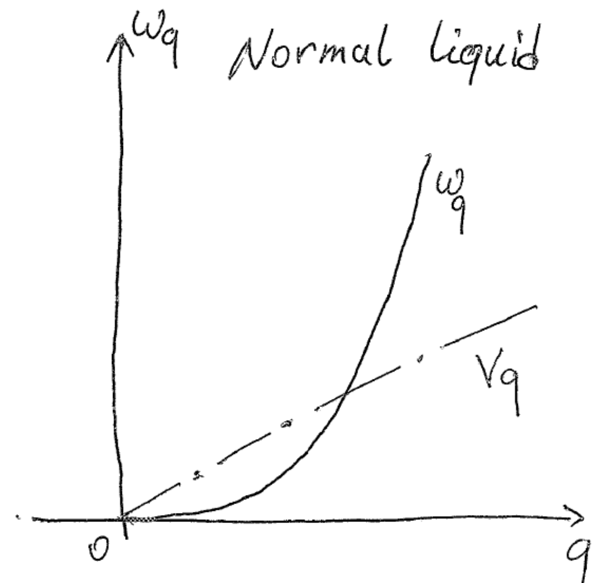
$$= \frac{1}{\sqrt{V}} \sum_{\vec{k}\vec{q}} \tilde{U}(\vec{q})\, \hat{\psi}^+_{\vec{k}+\vec{q}}\, \hat{\psi}_{\vec{q}}\, e^{-i\vec{q}\vec{V}t}$$



$$\sum_{\vec{k}} \hat{\psi}^+_{\vec{k}+\vec{q}}\, \hat{\psi}_{\vec{k}} \longrightarrow \sqrt{\bar{n}}\, \sqrt{S_q}\, (\hat{b}_q^+ + \hat{b}_{-q})$$

Rate of elementary excitations creation

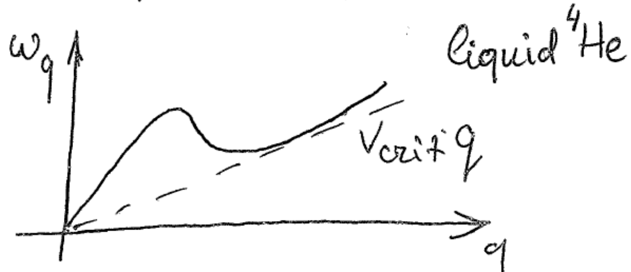
$$\Gamma = 2\pi\, \bar{n} \int \frac{d^3\vec{q}}{(2\pi)^3}\, |\tilde{U}(\vec{q})/\hbar|^2\, S_q\, \delta(\omega_q - \vec{V}\vec{q})$$



Superfluidity:

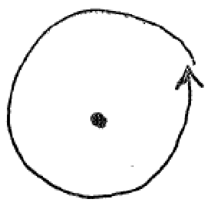
a moving body can dissipate energy only if $V > V_{crit}$

V_{crit} is given by ω_q



Vortices in BEC

Vortex line – topological defect



phase change by
$0, \pm 2\pi, \pm 4\pi$

$$\Psi = e^{is\varphi} |\Psi_0(\vec{r})|$$

$$s = \pm 1, \pm 2, \pm 3$$

$$\vec{v} = \frac{\hbar}{m} \frac{s}{\rho} \vec{e}_\varphi$$

$$\oint \vec{v}_s \, d\vec{l} = \int_{S_l} \mathrm{rot}\, \vec{v}_s \, d\vec{S} = 2\pi s \frac{\hbar}{m}$$

$$-\frac{\hbar^2}{2m} \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d\Psi_0}{d\rho} + \frac{\hbar^2 s^2}{2m\rho^2} \Psi_0 + g \Psi_0^3 = \mu \Psi_0 \qquad (*)$$

$$\rho \to 0 \qquad \Psi_0 \sim \rho^{|s|}$$

Energy of the vortex state (difference of the energy corresp. to Eq. (*) and the energy of the vortexless BEC):

$$E_v = L \pi \frac{\hbar^2}{m} n \ln\left(\frac{1{,}46 R}{\xi}\right) \qquad V = \pi R^2 L$$