1. Theory of elastic scattering: A general overview

$$-\frac{\hbar^2}{2m_{12}}\nabla^2\psi + V(\vec{r})\,\psi = \frac{\hbar^2}{2m_{12}}k^2\psi$$

Scattering potential   reduced mass $m_{12} = \dfrac{m_1 m_2}{m_1+m_2}$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right)$$

Asymptotically,

at $r\to\infty$ if $V(r)\to 0$ fast enough $k^2\psi = -\nabla^2\psi$

Axially-symmetric free motion:

$$\psi(r) = \sum_{\ell=0}^{\infty} P_\ell(\cos\vartheta)\left[c_\ell j_\ell(kr) + b_\ell y_\ell(kr)\right]$$

$$j_\ell(kr) = \sqrt{\frac{\pi}{2kr}}\,J_{\ell+1/2}(kr) \sim \frac{\sin(kr-\pi\ell/2)}{kr}, \quad r\to\infty$$

$$y_\ell(kr) = \sqrt{\frac{\pi}{2kr}}\,Y_{\ell+1/2}(kr) \sim -\frac{\cos(kr-\pi\ell/2)}{kr}$$

$$\psi\big|_{r\to\infty} \simeq e^{ikz} + \frac{f(\vartheta)}{r}e^{ikr} \qquad e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell(2\ell+1)P_\ell(\cos\vartheta)\,j_\ell(kr)$$

$f(\vartheta)$ = scattering amplitude

Differential cross-section $\dfrac{d\sigma}{d\Omega} = |f(\vartheta)|^2$ $d\Omega = 2\pi\sin\vartheta\, d\vartheta$

Scattering is elastic ➡ probability is conserved ➡ equal (by abs.value) coefficients in front of the incoming, ~ exp(ikr)/r, and outgoing, ~ exp(– ikr)/r, waves

$$\psi = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1) A_\ell P_\ell(\cos\vartheta) R_{k\ell}(kr)$$

$$R_{k\ell} \simeq \frac{1}{kr}\sin\left(kr - \frac{\pi\ell}{2} + \delta_\ell\right), \quad r \to \infty$$

the phase δ_l is determined by interaction at short distances

Scattered wave $\psi - e^{ikz} = \frac{f(\vartheta)}{r} e^{ikr}$ (asymptotically)

$$R_{k\ell} \simeq \frac{1}{2ikr}\left[(-i)^{\ell} e^{ikr+i\delta_\ell} - i^{\ell} e^{-ikr-i\delta_\ell}\right]$$

$$j_\ell(kr) \simeq \frac{1}{2ikr}\left[(-i)^{\ell} e^{ikr} - i^{\ell} e^{-ikr}\right]$$

To cancel terms $\sim \frac{1}{r} e^{ikr}$ we take $A_\ell = e^{i\delta_\ell}$

$$\psi = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1) e^{i\delta_\ell} P_\ell(\cos\vartheta) R_{k\ell}(kr) \simeq e^{ikz} + \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1) P_\ell(\cos\vartheta)\left(e^{2i\delta_\ell}-1\right)\frac{e^{ikr}}{r}$$

$$\boxed{f(\vartheta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left(e^{2i\delta_\ell}-1\right) P_\ell(\cos\vartheta)}$$

Scattering amplitude expressed via partial-wave phase shifts

l	0	1	2	3
Partial waves	s	p	d	f

$$\sigma = 2\pi \int_0^\pi d\vartheta \sin\vartheta \, |f(\vartheta)|^2 = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l$$

$$\int_0^\pi d\vartheta \sin\vartheta \, P_l(\cos\vartheta) \, P_{l'}(\cos\vartheta) = \frac{2}{2l+1} \delta_{ll'}$$

Collisions of identical particles (in the same spin state)

Permutation of particles 1↔2: $\vartheta \to \pi - \vartheta$, $\cos(\pi - \vartheta) = -\cos\vartheta$, $P_l(-X) = (-1)^l P_l(X)$

$$\frac{d\sigma}{d\Omega} = |f(\vartheta) \pm f(\pi - \vartheta)|$$

+ bosons (symmetric w.f.)
– fermions (antisymmetric w.f.)

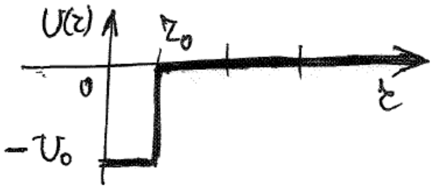
We cannot distinguish between two identical particles

But $0 < \vartheta < \pi/2$ (not $0 < \vartheta < \pi$)

$$\sigma = \frac{8\pi}{k^2} \sum_l (2l+1) \sin^2\delta_l$$

The sum is taken over even l (bosons) and odd l (fermions), respectively.

Exactly solvable model: scattering on a rectangular potential well



$$U(r) = \begin{cases} 0, & r \geq r_0 \\ -|U_0|, & r < r_0 \end{cases}$$

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty} \alpha_\ell P_\ell(\cos\vartheta) R_{k\ell}(kr)$$

$$R_{k\ell}(kr) = \begin{cases} b_\ell j_\ell(kr) + c_\ell y_\ell(kr), & r \geq r_0 \\ j_\ell(q_k r), & r < r_0 \end{cases}$$

regular at $r = 0$

$$q_k = \sqrt{k^2 + \frac{m|U_0|}{\hbar^2}}$$

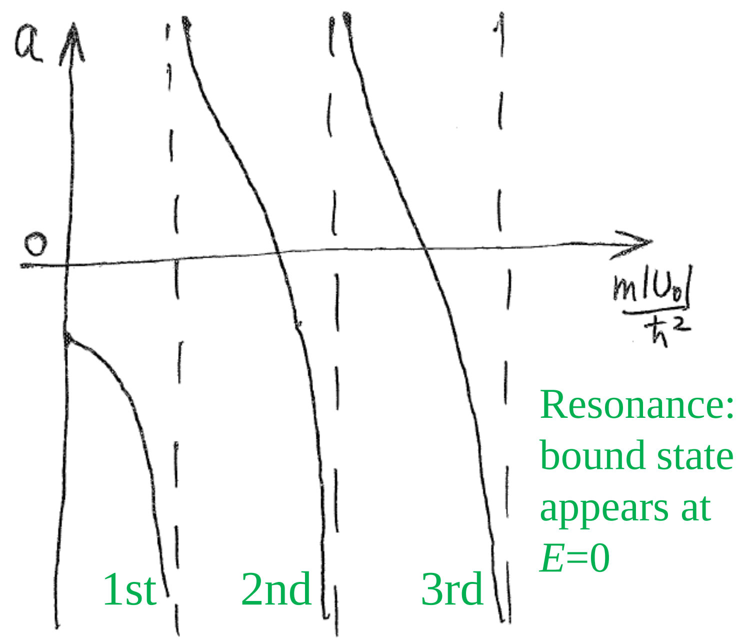
$R_{k\ell}$ and $\frac{\partial}{\partial r} R_{k\ell}$ are continuous at $r = r_0 \Rightarrow \delta_\ell$

s-wave scattering $(\ell = 0)$

$$R_{k\ell=0} = \begin{cases} \frac{\text{const}}{kr} \sin k(r-a), & r \geq r_0 \\ \frac{1}{kr} \sin q_k r \end{cases}$$

$$\delta_0 = -ka, \qquad \tan k(r_0 - a) = \frac{k}{q_k} \tan q_k r_0$$

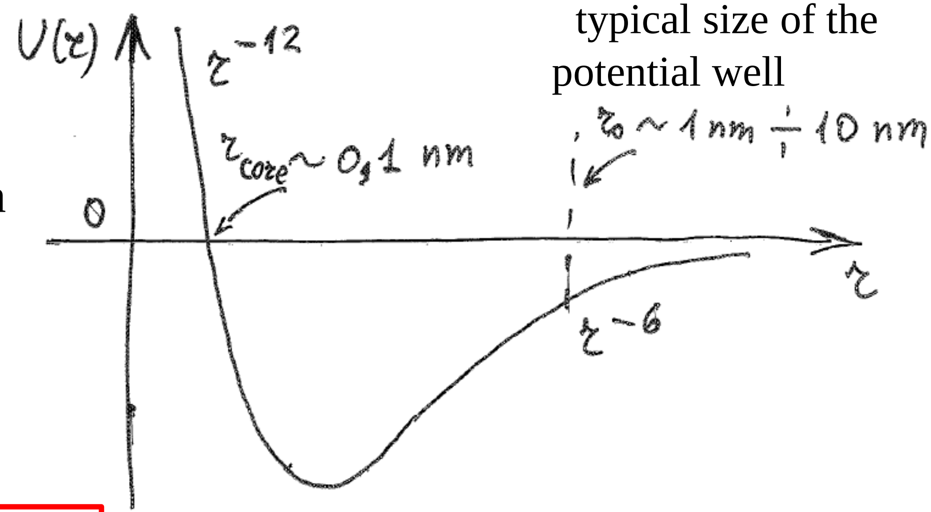
$$k r_0 \ll 1 \qquad \delta_\ell \sim k^{2\ell+1}$$



Far from resonances higher partial waves are negligible

The same conclusions hold for a general attractive potential

typical size of the potential well



far from resonnances (resonance occurs when a bound state appears at E=0 in a given partial wave):

$$\delta_l \sim k^{2l+1}, \qquad kr_0 \ll 1$$

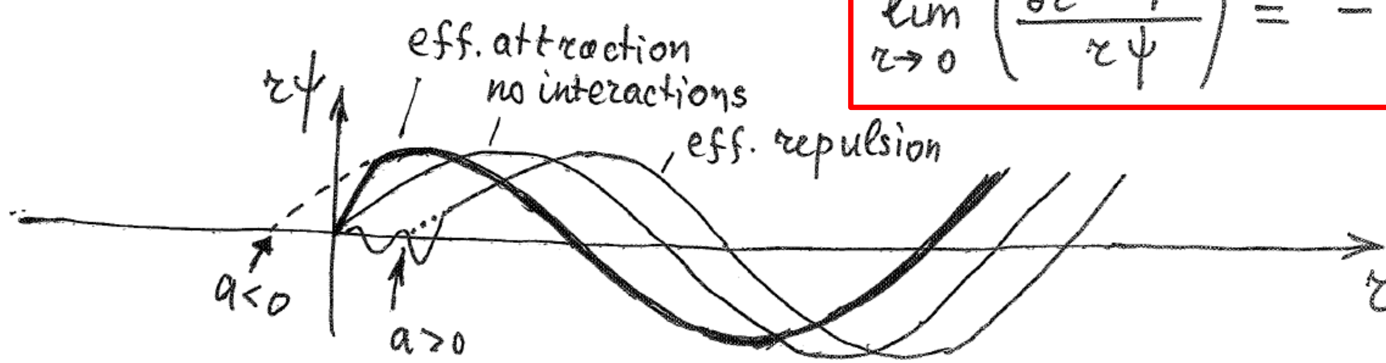
For bosons with the typical de Broglie wavelength >> 1 nm *s*-wave (isotropic) scattering dominates

$$\boxed{a = -\lim_{k\to 0} \frac{\delta_0}{k}}$$

s-wave scattering length

$$kr \ll 1: \qquad \psi(r) = \frac{\sin k(r-a)}{kr} \simeq 1 - \frac{a}{r}$$

$$\boxed{\lim_{r\to 0}\left(\frac{\frac{\partial}{\partial r}(r\psi)}{r\psi}\right) = -\frac{1}{a}}$$



Potential is substituted by a boundary condition

2. Macroscopic wave function of a BEC. Gross-Pitaevskii equation

The exact Hamiltonian of an *N*-particle system (bare Hamiltonian)

$$\hat{H}_B = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\nabla_j^2 + \sum_{j=1}^{N} V_{ext}(\vec{r}_j) + \sum_{j=1}^{N}\sum_{j'=1}^{j-1} v_{int}(\vec{r}_j - \vec{r}_{j'})$$

$$E\,\Psi_N(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = \hat{H}_B\,\Psi_N(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$$

Non-interacting system

$$\Psi_N(\vec{r}_1, \ldots \vec{r}_N) = (\text{normalization const}) \times \sum_{\text{permutations}} \psi_{\varepsilon_1}(\vec{r}_1)\,\psi_{\varepsilon_2}(\vec{r}_2)\ldots\psi_{\varepsilon_N}(\vec{r}_N)$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r})\right]\psi_{\varepsilon_j}(\vec{r}) = \varepsilon_j\,\psi_{\varepsilon_j}(\vec{r}) \qquad E = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_N$$

$$\int d^3\vec{r}\,|\psi_{\varepsilon_j}|^2 = 1$$

Ground state: all $\varepsilon_j = \varepsilon_0$, $E = N\varepsilon_0$, $\Psi_{N\,gr}(\vec{r}_1, \ldots, \vec{r}_N) = \prod_{j=1}^{N} \psi_{\varepsilon_0}(\vec{r}_j)$

$$\Psi_{N\,gr}(\vec{r}_1, \ldots, \vec{r}_N) \approx \prod_{j=1}^{N} \psi_0(\vec{r}_j) \prod_{j'=1}^{j-1} f(\vec{r}_j - \vec{r}_{j'})$$ Jastrow wave function

f describes interparticle correlations

Effective interaction (T-matrix) yields the correct ground state energy, but with a factorizable wave function:

$$\tilde{\Psi}_{N\,gr}(\vec{r}_1, \ldots, \vec{r}_N) = \prod_{j=1}^{N} \psi_0(\vec{r}_j)$$

Uniform gas: $V_{ext}(\vec{r}) \equiv 0$, we set periodic boundary conditions on the sides of a cube L^3

$$\Psi(\ldots, \vec{r}_j + L\vec{e}_x, \ldots) = \Psi(\ldots, \vec{r}_j, \ldots)$$

$$\frac{\partial}{\partial x_j}\Psi(\ldots, \vec{r}_j + L\vec{e}_x, \ldots) = \frac{\partial}{\partial x_j}\Psi(\ldots, \vec{r}_j, \ldots)$$

The same for y_j, z_j and all j

$$H_{eff} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\nabla_j^2 + \sum_{j=1}^{N} V_{ext}(\vec{r}_j) + \sum_{j=1}^{N}\sum_{j'=1}^{j-1}\tilde{T}(\vec{r}_j - \vec{r}_{j'})$$

$$k r_0 \ll 1 \qquad\qquad \tilde{T} = g\,\delta(\vec{r} - \vec{r}\,')$$

g is the same for the ground state and collectively excited states of the BEC.

The value of g is found from the comparison to the ground state of a uniform gas

Derivation of the GPE from the energy functional

$$W[\Psi_N] = \int d^3\vec{r}_1 \ldots \int d^3\vec{r}_N \, \Psi_N^* \hat{H}_B \Psi_N - E \int d^3\vec{r}_1 \ldots \int d^3\vec{r}_N \, \Psi_N^* \Psi_N$$

Lagrangian multiplier E has the meaning of the energy of the *N*-boson state

$$\int d^3\vec{r}_1 \ldots \int d^3\vec{r}_N \, \Psi_N^* \Psi_N = 1$$

Ψ_N and Ψ_N^* are varied independently

$$\frac{\delta}{\delta \Psi_N^*} W[\Psi_N] = 0 \quad \Rightarrow \quad E \Psi_N = \hat{H}_B \Psi_N$$

Factorizable state, effective Hamiltonian:

$$\tilde{\Psi}_N = \prod_{j=1}^{N} \psi_0(\vec{r}_j) \qquad \int d^3\vec{r} \, |\psi_0(\vec{r})|^2 = 1$$

$$\tilde{W}[\psi_0] = \int d^3\vec{r}_1 \ldots \int d^3\vec{r}_N \, \tilde{\Psi}_N^* \hat{H}_{eff} \tilde{\Psi}_N - \mu \int d^3\vec{r}_1 \ldots \int d^3\vec{r}_N \, \tilde{\Psi}_N^* \tilde{\Psi}_N$$

Lagrangian multiplier μ has the meaning of the chem.potential $\mu = \frac{\partial E_{gr}}{\partial N}$

$$\frac{\delta}{\delta \psi_0^*} \tilde{W}[\psi_0] = 0 \quad \Rightarrow \quad \mu \psi_0 = -\frac{\hbar^2}{2m} \nabla^2 \psi_0 + V_{ext}(\vec{r}) \psi_0 + g(N-1)|\psi_0|^2 \psi_0$$

$$N - 1 \approx N \gg 1 \qquad \sqrt{N} \psi_0 \equiv \Psi \quad , \quad \int d^3\vec{r} \, |\Psi|^2 = N$$

$$\mu \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V_{ext}(\vec{r}) \Psi + g |\Psi|^2 \Psi$$

Uniform gas: $\Psi = \sqrt{\bar{n}} \equiv \sqrt{\frac{N}{\mathcal{V}}}$ $\qquad \mu = g\bar{n}$ $\qquad \mathcal{V} = L^3$

$$\Psi_N(\vec{r}_1, \dots, \vec{r}_N) = \frac{1}{\sqrt{\mathcal{V}^N}} \prod_{j'<j} f(\vec{r}_j - \vec{r}_{j'})$$

$$E = \int d^3\vec{r}_1 \dots \int d^3\vec{r}_N\, \Psi_N^* \hat{H}_B \Psi_N \approx \frac{N(N-1)}{2}\varepsilon_{pair} \approx \frac{N^2}{2}\varepsilon_{pair}$$

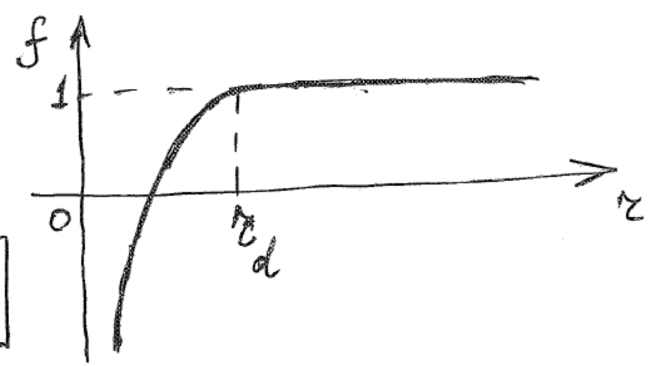
Cluster expansion

Lowest-order constrained variational (LOCV) approach

$$\varepsilon_{pair} = \frac{\int d^3\vec{r}\, f^* \left(-\frac{\hbar^2}{m}\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right) f}{\int d^3\vec{r}\, |f|^2}$$

$$-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} f = q^2 f \qquad \frac{\frac{\partial}{\partial r}(rf)}{rf} \to -\frac{1}{a}, \quad r \to 0 \qquad \frac{\partial f}{\partial r} = 0, \quad r = r_d$$

$$r_d \sim \bar{n}^{-1/3}, \quad \text{e.g.,} \quad \frac{4\pi}{3} r_d^3 = \bar{n}^{-1}$$



normalization $\quad f \equiv 1 \quad r > r_d$

$$\int d^3\vec{r}\, |f|^2 = \mathcal{V}\left[1 - \mathcal{O}\left(\frac{r_d^3}{\mathcal{V}}\right)\right] = \mathcal{V}\left[1 - \mathcal{O}\left(\frac{1}{N}\right)\right]$$

$$\varepsilon_{pair} \simeq \frac{\hbar^2 q^2}{m\mathcal{V}} \cdot 4\pi \int_0^{r_d} dr \cdot r^2 |f|^2$$

$$f = \frac{r_d}{r}\,\frac{\sin q(r-b)}{\sin q(r_d-b)}$$

Boundary conditions at $r \to 0$:

$$\tan qb = qa$$

$r = r_d$:

$$\frac{(qr_d)^{-1}\tan qr_d - 1}{1 + qr_d \tan qr_d} = \frac{a}{r_d}$$

Weak interactions $a/r_d \ll 1$, i.e., $\bar{n}a^3 \ll 1$

$$\tan qr_d \approx qr_d + \frac{1}{3}(qr_d)^3 \implies q^2 \approx \frac{3a}{r_d^3}$$

$$4\pi\int_0^{r_d} dr \cdot r^2 |f|^2 = \frac{4\pi}{3}r_d^3\left[1 + O\left(\frac{a}{r_d}\right)\right] \approx \frac{4\pi}{3}r_d^3$$

$$\varepsilon_{pair} = \frac{4\pi\hbar^2 a}{mV}$$

$$E_{gr} = \frac{N^2}{2}\varepsilon_{pair} = \frac{2\pi\hbar^2 N^2 a}{mV} \qquad \mu = \left(\frac{\partial E_{gr}}{\partial N}\right)_V = \frac{4\pi\hbar^2 \bar{n} a}{m}$$

$$\boxed{g = \frac{4\pi\hbar^2 a}{m}}$$

GPE can be written also for a non-stationary (time-dependent) case

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + V_{ext}(\vec{r})\Psi + \frac{4\pi\hbar^2 a}{m}|\Psi|^2\Psi$$

The scheme to linearize the time-dependent problem

$$\mu\Psi_0 = -\frac{\hbar^2}{2m}\nabla^2\Psi_0 + V_{ext}(\vec{r})\Psi_0 + \frac{4\pi\hbar^2 a}{m}|\Psi_0|^2\Psi_0$$

$\arg\Psi_0 = \text{const} \Rightarrow \Psi_0$ may be chosen <u>real</u>

$$\Psi = (\Psi_0 + \delta\psi)e^{-i\mu t/\hbar} \qquad |\Psi|^2\Psi \approx \Psi_0^2 e^{-i\mu t/\hbar}(\Psi_0 + 2\delta\psi + \delta\psi^*)$$

$$i\hbar\frac{\partial\delta\psi}{\partial t} = \hat{\mathcal{L}}_{HF}\,\delta\psi + \frac{4\pi\hbar^2 a}{m}n_0(\vec{r})\delta\psi^*$$

$$\hat{\mathcal{L}}_{HF} = -\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) + \frac{8\pi\hbar^2 a}{m}n_0(\vec{r}) - \mu \qquad n_0(\vec{r}) = \Psi_0^2(\vec{r})$$

$$\delta\psi = u(\vec{r})e^{-i\omega t} - v^*(\vec{r})e^{i\omega t}$$

$$\left.\begin{aligned} \hbar\omega u &= \hat{\mathcal{L}}_{HF}u - \frac{4\pi\hbar^2 a\, n_0(\vec{r})}{m}v \\ -\hbar\omega v &= \hat{\mathcal{L}}_{HF}v - \frac{4\pi\hbar^2 a\, n_0(\vec{r})}{m}u \end{aligned}\right\}$$ Bogoliubov – de Gennes equations

Uniform gas

$$n_0 = \text{const}, \qquad V_{ext} \equiv 0 \qquad u(\vec{r}) = u_k \frac{e^{i\vec{k}\vec{r}}}{\sqrt{V}} \qquad v(\vec{r}) = v_k \frac{e^{i\vec{k}\vec{r}}}{\sqrt{V}}$$

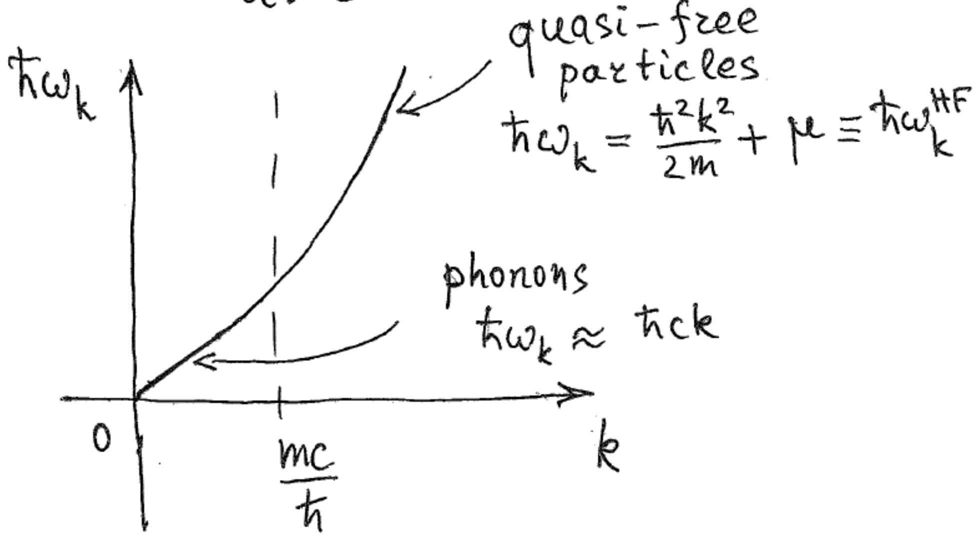
$$\hbar\omega_k u_k = \left(\frac{\hbar^2k^2}{2m} + \mu\right) u_k - \mu v_k$$

$$-\hbar\omega_k v_k = \left(\frac{\hbar^2k^2}{2m} + \mu\right) v_k - \mu u_k$$

$$\mu = \frac{4\pi\hbar^2 a n_0}{m} \equiv mc^2$$

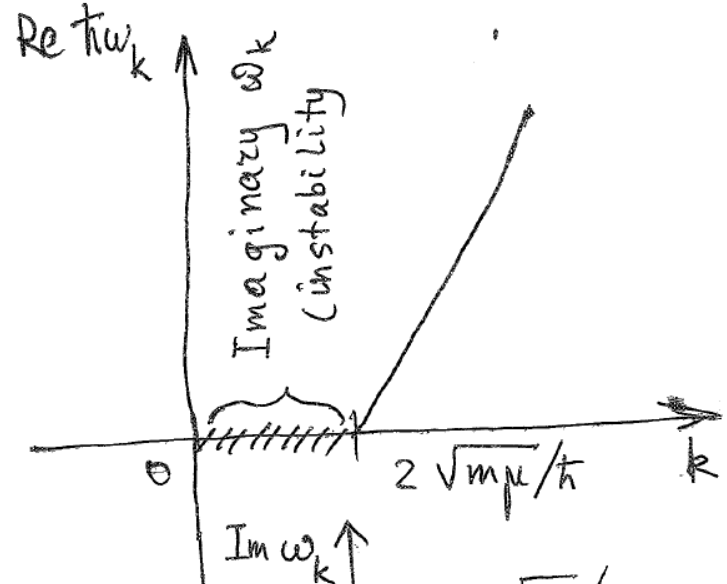
$$\hbar\omega_k = \sqrt{\frac{\hbar^2k^2}{2m}\left(\frac{\hbar^2k^2}{2m} + 2\mu\right)}$$ Bogoliubov dispersion law

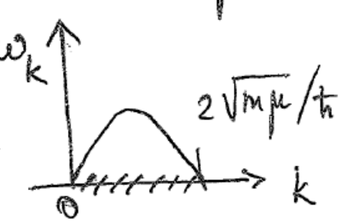
$a > 0$



effective repulsion

$a < 0$





effective attractionion

3. BEC description in the second-quantization formalism

A general case of a system of identical particles with pairwise interactions:

$$\hat{H} = \sum_{j=1}^{N} \hat{H}^{(1)}_j + \sum_{j=1}^{N} \sum_{j'=1}^{j-1} \mathcal{U}^{(2)}(\vec{r}_j - \vec{r}_{j'})$$

$$\hat{H} = \int d^3\vec{r}\ \hat{\Psi}^+(\vec{r}) \left[\hat{H}^{(1)} + \frac{1}{2} \int d^3\vec{r}\,'\ \hat{\Psi}^+(\vec{r}\,') \mathcal{U}^{(2)}(\vec{r} - \vec{r}\,') \hat{\Psi}(\vec{r}\,') \right] \hat{\Psi}(\vec{r})$$

Bosonic field: $\left[\hat{\Psi}(\vec{r}), \hat{\Psi}^+(\vec{r}\,')\right] = \delta(\vec{r} - \vec{r}\,')$ $\quad \left[\hat{\Psi}(\vec{r}), \hat{\Psi}(\vec{r}\,')\right] = \left[\hat{\Psi}^+(\vec{r}\,'), \hat{\Psi}^+(\vec{r})\right] = 0$

$$\hat{\Psi} = \sum_n \hat{a}_n \varphi_n(\vec{r}) \qquad \int d^3\vec{r}\ \varphi_n^*(\vec{r}) \varphi_{n'}(\vec{r}) = \delta_{nn'}$$

$$\sum_n \varphi_n^*(\vec{r}\,') \varphi_n(\vec{r}) = \delta(\vec{r} - \vec{r}\,')$$

$$[\hat{a}_n, \hat{a}^+_{n'}] = \delta_{nn'} \qquad [\hat{a}_n, \hat{a}_{n'}] = [\hat{a}^+_{n'}, \hat{a}^+_n] = 0$$

$$\hat{a}_n |\mathcal{N}_n\rangle_n = \sqrt{\mathcal{N}_n}\, |\mathcal{N}_n - 1\rangle_n \qquad \hat{a}_n |0\rangle_n = 0 \qquad \hat{a}^+_n |\mathcal{N}_n\rangle_n = \sqrt{\mathcal{N}_n + 1}\, |\mathcal{N}_n + 1\rangle_n$$

$$\hat{H}'_{eff} = \int d^3\vec{r} \left\{ \hat{\Psi}^+ \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + V_{ext}(\vec{r}) \right] \hat{\Psi} + \frac{2\pi\hbar^2 a}{m} \hat{\Psi}^+ \hat{\Psi}^+ \hat{\Psi} \hat{\Psi} \right\}$$

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi} = -\left[\hat{H}'_{eff}, \hat{\Psi} \right] = -\frac{\hbar^2}{2m}\nabla^2 \hat{\Psi} - \mu \hat{\Psi} + V_{ext}(\vec{r}) \hat{\Psi} + \frac{4\pi\hbar^2 a}{m} \hat{\Psi}^+ \hat{\Psi} \hat{\Psi}$$

Bose-Einstein condensation in the single-particle ground state:

$$[\hat{a}_0, \hat{a}_0^+] = 1 \ll \langle a_0^+ a_0 \rangle = N_c$$

in a grand canonical ensemble at $T \ll T_{crit}$ $N_c = N$

Symmetry-breaking assumption $\hat{a}_0, \hat{a}_0^+ \to \sqrt{N}$ $\langle \hat{\Psi} \rangle = \Psi_0 \neq 0$

can be chosen real

$$\mu \Psi_0 = -\frac{\hbar^2}{2m}\nabla^2 \Psi_0 + V_{ext}(\vec{r})\Psi_0 + \frac{4\pi\hbar^2 a}{m}|\Psi_0|^2 \Psi_0$$

$$i\hbar \frac{\partial}{\partial t}\delta\hat{\psi} = \hat{\mathcal{L}}_{HF}\delta\hat{\psi} + \frac{4\pi\hbar^2}{m} a\, n_0(\vec{r})\,\delta\hat{\psi}^+ \qquad \hat{\Psi} = \Psi_0 + \delta\hat{\psi}$$

$$\delta\hat{\psi} = \sum_{n\neq 0}\left[u_n(\vec{r})e^{-i\omega_n t}\hat{b}_n - v_n^*(\vec{r})e^{i\omega_n t}\hat{b}_n^+\right]$$

$\hat{b}_n$ — annihilation operator of an elementary excitation (Bogoliubov quasiparticle)

$$[\hat{b}_n, \hat{b}_{n'}^+] = \delta_{nn'} \qquad [\hat{b}_n, \hat{b}_{n'}] = [\hat{b}_{n'}^+, \hat{b}_n^+] = 0$$

u_n and v_n – solutions of Bogoliubov – de Gennes eqs.

$$\int d^3\vec{r}\, \Psi_0^*(u_n - v_n) = 0 \qquad \int d^3\vec{r}\left[u_n^* u_{n'} - v_n^* v_{n'}\right] = \delta_{nn'}$$

Uniform gas:

$$\delta\hat{\psi} = \sum_{k\neq 0}\left(u_k \hat{b}_k(t) - v_k \hat{b}_{-k}^+(t)\right)\frac{e^{i\vec{k}\vec{r}}}{\sqrt{V}}$$

$$\hat{b}_k(t) = \hat{b}_k(0)\, e^{-i\omega_k t}$$

Bogoliubov coefficients:

$$\left.\begin{matrix} u_k \\ v_k \end{matrix}\right\} = \sqrt{\frac{\hbar\omega_k^{HF}}{2\hbar\omega_k} \pm \frac{1}{2}} \qquad \hbar\omega_k^{HF} = \frac{\hbar^2 k^2}{2m} + \mu$$

Number of non-condensate atoms:

$$N_{nc} = \int d^3\vec{r} \langle \delta\hat{\psi}^+ \delta\hat{\psi} \rangle = \sum_{k \neq 0} \langle \hat{a}_k^+ \hat{a}_k \rangle$$

Bogoliubov approximation: $N_{nc} \ll N$

$N_{nc} \sim N$ Hartree-Fock-Bogoliubov-Popov approx.

$$\langle b_n b_{n'} \rangle = \langle b_n^+ b_{n'}^+ \rangle = 0 \qquad \langle b_n^+ b_{n'} \rangle = \delta_{nn'} \langle b_n^+ b_n \rangle$$

$$\langle b_n^+ b_n \rangle = \frac{1}{\exp\left(\frac{\hbar\omega_k}{k_B T}\right) - 1}$$

Chemical potential for elementary excitations is <u>zero</u>.

$$N_{nc} = \int d^3\vec{r} \sum_{n \neq 0} \left[(|u_n|^2 + |v_n|^2) \langle b_n^+ b_n \rangle + |v_n|^2 \right]$$

Thermal depletion (vanishes at $T \to 0$)

Quantum depletion ($\neq 0$ at $T = 0$)

$$N_{nc}\Big|_{T=0} = \mathcal{V} \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\frac{\hbar\omega_k^{HF}}{2\hbar\omega_k} - 1 \right) = \frac{8}{3\sqrt{\pi}} N \sqrt{\bar{n} a^3} \qquad \bar{n} = \frac{N}{\mathcal{V}}$$

Linearized Hamiltonian

$$\hat{H}'_{eff} = E'_{gr} + \sum_{n\neq 0} \hbar\omega_n \hat{b}_n^+ \hat{b}_n$$

$$E_{gr} = E_{gr}^{(0)} + \delta E_{gr} \qquad \frac{\partial E_{gr}^{(0)}}{\partial N} = \mu$$

$\delta E_{gr} =$ correction to the ground state energy (2nd order in a)

„Naive“ expression:

$$\delta E_{gr} = \frac{1}{2}\sum_{n\neq 0} \hbar\omega_n - \frac{1}{2}\sum_{n\neq 0} \hbar\omega_n^{HF}$$

Uniform gas: $$\delta E_{gr} = -\frac{\mathcal{V}}{(2\pi)^2}\int_0^\infty dk \cdot k^2 \frac{\mu^2}{\hbar\omega_k + \hbar\omega_k^{HF}}$$ – diverges!

Removing the divergency by renormalization of the coupling constant

$$\tilde{T}(\vec{r}-\vec{r}\,') = \frac{4\pi\hbar^2 a}{m}\left(1 + 4\pi a \int \frac{d^3\vec{k}'}{k'^2}\right)\delta(\vec{r}-\vec{r}\,')$$

$$E_{gr} = E_{gr}^{(0)} + \delta E_{gr} = \frac{2\pi\hbar^2 a N^2}{m\mathcal{V}}\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\bar{n}a^3}\right)$$

$$\mu + \delta\mu = \left(\frac{\partial E_{gr}}{\partial N}\right)_{\mathcal{V}} = \frac{4\pi\hbar^2 a\bar{n}}{m}\left(1 + \frac{32}{3\sqrt{\pi}}\sqrt{\bar{n}a^3}\right)$$

Local density approximation (LDA) for a trapped BEC

$$n_0(\vec{r}) = |\Psi_0|^2 \qquad N_{nc} = \frac{8}{3\sqrt{\pi}} \int d^3\vec{r}\, n_0(\vec{r}) \sqrt{n_0(\vec{r}) a^3}$$

Size of the trapped BEC must be much smaller than a certain length

$T \ll T_{crit}$ ⇨ the relevant length is the healing length $\xi_h = \frac{\hbar}{mc}$

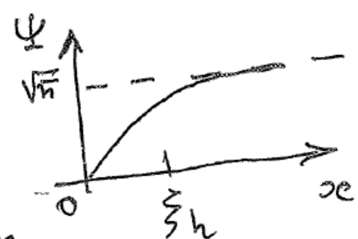
BEC in a half-space $x > 0$

$$\mu \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi + \frac{4\pi\hbar^2 a}{m} |\Psi|^2 \Psi \qquad \Psi|_{x=0} = 0 \qquad \Psi|_{x\to+\infty} \to \sqrt{\bar{n}}$$

$$\Psi = \sqrt{\bar{n}}\, \phi(\bar{x}) \quad , \bar{x} = x/\xi_h = mcx/\hbar$$

$$\phi = -\frac{1}{2} \frac{d^2}{dx^2} \phi + |\phi|^2 \phi \qquad \phi|_{x=0} = 0 \qquad \phi|_{x\to+\infty} = 1$$

$$\phi = \tanh \bar{x} \qquad \Psi = \sqrt{\bar{n}} \tanh x/\xi_h$$



An impurity atom in a BEC (polaron)